NN scattering problem in EFT reformulated

Jambul Gegelia

in collaboration with Evgeny Epelbaum

Ruhr-Universität Bochum

The 7th International Workshop on Chiral Dynamics
August 6 - 10, 2012, Jefferson Lab, Newport News, VA

Outline

- Stating the problem;
- New approach;
- Phase shifts;
- Summary;

Few-nucleon sector of BChPT first considered in

S. Weinberg, Phys. Lett. B **251**, 288 (1990).

Weinberg's program for few-nucleon systems encounters a non-trivial problem:

The NN potential of chiral EFT is non-renormalizable in the traditional sense already at LO.

Renormalization of the solution to the LS equation requires an infinite number of counter-terms.

This problem has been addressed as the inconsistency of Weinberg's approach.

An alternative power counting scheme (KSW):

D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B **424**, 390 (1998)

In KSW: OPE potential is of NLO $\Rightarrow$ LO equation is renormalizable perturbatively as well as non-perturbatively.

Corrections are treated perturbatively. As a consequence, no "consistency problem."

However, perturbative series do not converge in the KSW approach.

Expansion of nuclear forces about the chiral limit:

S. R. Beane, P. F. Bedaque, M. J. Savage, and U. van Kolck, Nucl. Phys. **A700**, 377 (2002).

This expansion is formally consistent and is equivalent to the KSW/Weinberg power counting in the ${}^1S_0$ channel/the ${}^3S_1 - {}^3D_1$ coupled channels.

Higher partial waves:

A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C **72**, 054006 (2005).

Perturbative treatment of the OPE potential is not sufficient for a finite number of partial waves.

Iterated OPE potential produces cutoff dependence in all waves where the tensor force is attractive.

Include counter-terms in each of these partial waves.

A modified KSW approach:

S. R. Beane, D. B. Kaplan and A. Vuorinen, Phys. Rev. C **80**, 011001 (2009).

It uses a more adequate renormalization scheme to improve the convergence of the perturbative series.

We have implemented the new KSW approach.

Problems with convergence seem to persist.

There exists an alternative point of view that the "consistency problem" of Weinberg's approach is irrelevant if the cutoff Λ is kept finite.

Taking large values of Λ without including all relevant counter-terms is not a legitimate procedure. Instead one has to choose $\Lambda \sim 1\,\mathrm{GeV}$.

Here we present a modified version of Weinberg's approach: free of the "consistency problem".

Similar to the "semi-relativistic" scheme of

D. Djukanovic, J. Gegelia, S. Scherer and M. R. Schindler, Few Body Syst. **41**, 141 (2007).

Difference: Time-ordered perturbation theory; Different treatment of the two-nucleon propagator.

- We start with manifestly Lorentz invariant effective Lagrangian applying time-ordered PT.

- LO amplitude is obtained by solving an integral equation; corrections are calculated perturbatively.

- All divergences are absorbed in the redefinition of parameters of the potential at given order.

- OPE potential is treated non-perturbatively.

- In $^3P_0$ channels the LO integral equation does not have a unique solution.

  We solve this problem by including a counter-term at LO.

Effective Lagrangian of pions and nucleons:

$$\mathcal{L}_{\mathsf{eff}} = \mathcal{L}_{\pi} + \mathcal{L}_{\pi\mathsf{N}} + \mathcal{L}_{\mathsf{NN}} + \cdots .$$

Use the time-ordered perturbation theory:

S. Weinberg, Phys. Rev. **150**, 1313 (1966).

Decompose the nominator of fermion propagator as

$$\not p + m = 2\, m\, P_{+} + (\not p - m \not v)\,,$$

where $P_{+} = \frac{1+\not v}{2}$ with $v = (1,0,0,0)$, and identify the second term as of higher order.

For NN scattering, define the two-nucleon-irreducible diagrams as the effective potential V.

NN vertex function T satisfies an integral equation

$$T = V + V\,G\,T\,.$$

Here G is the two-nucleon propagator.

Substituting the expansions of V, G and T in a small parameter we solve T order by order.

At leading order:

$$T_0 = V_0 + V_0\,G_0\,T_0\,.$$

Corrections are calculated perturbatively.

The physical amplitude using LSZ formalism

$$\tilde{T} = Z_\psi^2\, \bar{u}_3 \bar{u}_4\, T\, u_1 u_2,$$

where Z_ψ is the residue of the propagator and

$$\begin{aligned} u(p) &= \left(1 + \frac{\not{p} - p\cdot v}{m + p\cdot v}\right)\frac{1+\not{v}}{2} u(p) = u_0 + u_1\,, \\ \bar{u}(p) &= \bar{u}(p)\frac{1+\not{v}}{2}\left(1 + \frac{\not{p} - p\cdot v}{m + p\cdot v}\right) = \bar{u}_0 + \bar{u}_1\,, \end{aligned}$$

are Dirac spinors;

$$\begin{aligned} u_0(p) &= P_+\, u(p)\,, \\ \bar{u}_0(p) &= \bar{u}(p)\, P_+\,, \end{aligned}$$

LO equation for the physical amplitude

$$\tilde{T}_0 = \tilde{V}_0 + \tilde{V}_0\, G\, \tilde{T}_0\,,$$

with $\tilde{V}_0 = P_+ P_+\, V_0\, P_+ P_+$, the projected potential.

Equation for LO scattering amplitude in COM frame

$$\tilde{T}_0\left(\vec{p}\,',\vec{p}\right) = \tilde{V}_0\left(\vec{p}\,',\vec{p}\right) - \frac{m^2}{2}\int \frac{d^3\vec{k}}{(2\,\pi)^3}\,\tilde{V}_0\left(\vec{p}\,',\vec{k}\right) \times \frac{1}{\left(\vec{k}^2+m^2\right)\,\left(p_0-\sqrt{\vec{k}^2+m^2}+i\,\epsilon\right)}\,\tilde{T}_0\left(\vec{k},\vec{p}\right),$$

where the LO effective potential $\tilde{V}_0 = \tilde{V}_{0,C} + \tilde{V}_{0,\pi}$ with

$$\tilde{V}_{0,C} = -C_S - C_T\,\vec{\sigma}_1\cdot\vec{\sigma}_2$$

and

$$\tilde{V}_{0\pi}\left(\vec{p}\,',\vec{p}\right) = -\frac{g_A^2}{4\,F^2}\sum_{a=1}^{3}\frac{\vec{\sigma}_1\cdot(\vec{p}-\vec{p}\,')\,\vec{\sigma}_2\cdot(\vec{p}-\vec{p}\,')}{(\vec{p}-\vec{p}\,')^2+M^2}\quad .$$

LO partial wave equations:

$$\begin{aligned} T^{sj}_{l'l}\left(p',p\right) &= V^{sj}_{l'l}\left(p',p\right) \\ &+ \frac{m^2}{2}\sum_{l''}\int_0^\infty \frac{dk\,k^2}{(2\,\pi)^3}\,\frac{V^{sj}_{l'l''}\left(p',k\right)\,T^{sj}_{l''l}\left(k,p\right)}{\left(k^2+m^2\right)\left(p_0-\sqrt{k^2+m^2}+i\,\epsilon\right)}, \end{aligned}$$

where $V^{sj}_{l'l}\left(p',p\right)$ is the partial wave projected NN potential and $p=|\vec{p}|$, $p'=|\vec{p}\,'|$.

- LO equation has a milder UV behavior than the corresponding LS equation.

- Iterations of LO equation generate diagrams which only contain overall logarithmic divergences.

- These divergences are absorbed in the redefinition of the parameters of the LO potential $\Rightarrow$ LO equation is perturbatively renormalizable.

In non-perturbative regime: approximate the two-nucleon propagator for $k \to \infty$ by

$$\frac{p_0 + \sqrt{k^2 + m^2}}{\left(k^2 + m^2\right)\left(p^2 - k^2 + i\,\epsilon\right)} \to \frac{1}{k\left(p^2 - k^2 + i\,\epsilon\right)},$$

and obtain

$$T^{sj}_{l'l}\left(p', p\right) = V^{sj}_{l'l}\left(p', p\right) + \frac{m^2}{2} \sum_{l''} \int_0^\infty \frac{dk\,k}{(2\,\pi)^3} \frac{V^{sj}_{l'l''}\left(p', k\right)\, T^{sj}_{l''l}\left(k, p\right)}{p^2 - k^2 + i\,\epsilon}.$$

It has the form of the partial wave LS equation in three space-time dimensions.

Corresponding OPE potential behaves as $\sim \frac{1}{r^2}$ for $r \to 0$.

More singular $\sim \frac{1}{r^3}$ UV behavior of the potential in non-relativistic (HBChPT) is an artifact of that formulation.

Guaranteed correct non-relativistic expansion of physical quantities is obtained by performing Lorentz invariant calculation and after performing the expansion.

In HB approach one expands on the level of the Lagrangian, however includes additional compensating terms.

Any mismatch between the two expansions has to be attributed to the shortcomings of the HB approach.

Modified UV behavior in heavy baryon expansion is easily seen in the following integral, contributing in the scattering amplitude of two scalar "baryons" with $P = (2\sqrt{m^2+p^2}, \vec{0}\,)$

$$
\begin{aligned}
I &= \frac{4\,i}{(2\,\pi)^4}\int \frac{d^4k\,\theta(\Lambda-|\vec{k}|)}{\left[k^2-m^2+i\,0^+\right]\left[(P-k)^2-m^2+i\,0^+\right]} \\
&= \frac{1}{(2\,\pi)^3}\int \frac{d^3\vec{k}\,\theta(\Lambda-|\vec{k}|)}{\left[\vec{k}^2+m^2\right]\left[p_0-\sqrt{\vec{k}^2+m^2}+i\,0^+\right]} \\
&- \frac{1}{(2\,\pi)^3}\int \frac{d^3\vec{k}\,\theta(\Lambda-|\vec{k}|)}{\left[\vec{k}^2+m^2\right]\left[p_0+\sqrt{\vec{k}^2+m^2}\right]}. \\
&= I_1+I_2\,.
\end{aligned}
$$

I_1 and I_2 correspond to diagrams of the old-fashioned time-ordered perturbation theory.

The results of integrals for $\Lambda > p$:

$$
\begin{aligned}
I_1 &= \frac{p \ln \frac{\Lambda\sqrt{m^2+p^2}+p\sqrt{\Lambda^2+m^2}}{\Lambda\sqrt{\Lambda^2+m^2}-p\sqrt{m^2+p^2}}}{4\pi^2\sqrt{m^2+p^2}} - \frac{\ln \frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}}{2\pi^2} \\
&+ \frac{p \tanh^{-1}\frac{p}{\Lambda} - m \tan^{-1}\frac{\Lambda}{m}}{2\pi^2\sqrt{m^2+p^2}} - \frac{ip}{2\pi\sqrt{m^2+p^2}}, \\
I_2 &= \frac{p \ln \frac{\Lambda\sqrt{m^2+p^2}+p\sqrt{\Lambda^2+m^2}}{m\sqrt{\Lambda^2-p^2}}}{2\pi^2\sqrt{m^2+p^2}} - \frac{\ln \frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}}{2\pi^2} \\
&+ \frac{m \tan^{-1}\frac{\Lambda}{m} - p \tanh^{-1}\frac{p}{\Lambda}}{2\pi^2\sqrt{m^2+p^2}}, \\
I &= \frac{p \ln \frac{\Lambda\sqrt{m^2+p^2}+p\sqrt{\Lambda^2+m^2}}{m\sqrt{\Lambda^2-p^2}}}{\pi^2\sqrt{m^2+p^2}} - \frac{\ln \frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}}{\pi^2} - \frac{ip}{2\pi\sqrt{m^2+p^2}}.
\end{aligned}
$$

Expand first in Λ and subsequently in $1/m$:

$$
\begin{aligned}
I_1^1 &= -\frac{ip}{2\pi m} + \frac{m}{2\pi^2\Lambda} - \frac{\ln\frac{\Lambda}{m}}{2\pi^2} - \frac{\pi + \ln 4}{4\pi^2} + \frac{p^2}{4\pi^2\Lambda m} + \mathcal{O}\left(\frac{1}{\Lambda^2}, \frac{1}{m^2}\right), \\
I_2^1 &= -\frac{m}{2\pi^2\Lambda} - \frac{p^2}{4\pi^2\Lambda m} - \frac{\ln\frac{\Lambda}{m}}{2\pi^2} + \frac{\pi - \ln 4}{4\pi^2} + \mathcal{O}\left(\frac{1}{\Lambda^2}, \frac{1}{m^2}\right), \\
I^1 &= -\frac{ip}{2\pi m} - \frac{\ln\frac{\Lambda}{m}}{\pi^2} - \frac{\ln 2}{\pi^2} + \mathcal{O}\left(\frac{1}{\Lambda^2}, \frac{1}{m^2}\right).
\end{aligned}
$$

The one-loop integral I is logarithmically divergent.

Expand first in m and after in Λ about ∞:

$$
\begin{aligned}
I_1^2 &= -\frac{ip}{2\pi m} - \frac{\Lambda}{\pi^2 m} + \frac{p^2}{\pi^2 \Lambda m} + \mathcal{O}\left(\frac{1}{m^2}, \frac{1}{\Lambda^2}\right), \\
I_2^2 &= \mathcal{O}\left(\frac{1}{m^2}, \frac{1}{\Lambda^2}\right), \\
I^2 &= -\frac{ip}{2\pi m} - \frac{\Lambda}{\pi^2 m} + \frac{p^2}{\pi^2 \Lambda m} + \mathcal{O}\left(\frac{1}{m^2}, \frac{1}{\Lambda^2}\right).
\end{aligned}
$$

Two expansions are not commutative:

HB approach corresponds to terms $\sim 1/m$ in I_1^2.

HB expansion leads to qualitatively different UV behavior.

This is not a problem in perturbative calculations.

For non-perturbative equations, one needs to include contributions of an infinite number of terms.

Otherwise, in HB approach one is not allowed to take Λ much bigger than the nucleon mass m.

Integral I_1 of time-ordered PT has the same UV behavior as the integral I, guaranteeing correct qualitative UV behavior in our new approach.

In time-ordered PT the nucleon mass DOES NOT play the role of the cutoff!

Due to radial repulsion integral equations have unique solutions for all partial waves except ${}^3P_0$.

In ${}^3P_0$ channel the LO equation does not have an unique solution. This is the same behavior as in S-TM equation:

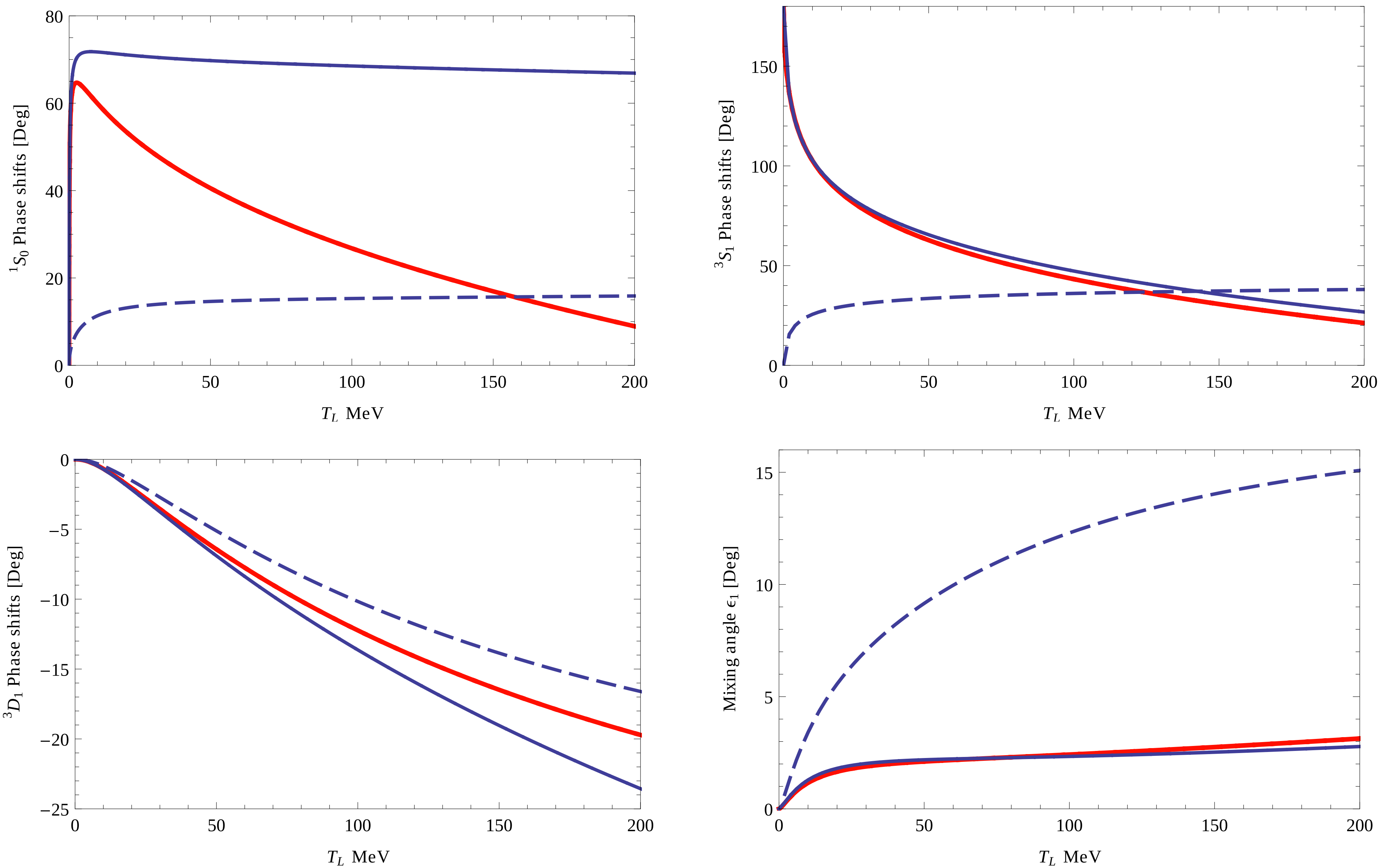
G. V. Skornyakov and Ter-Martirosyan, Sov. Phys. JETP **4**, 648 (1957).

Analogously to

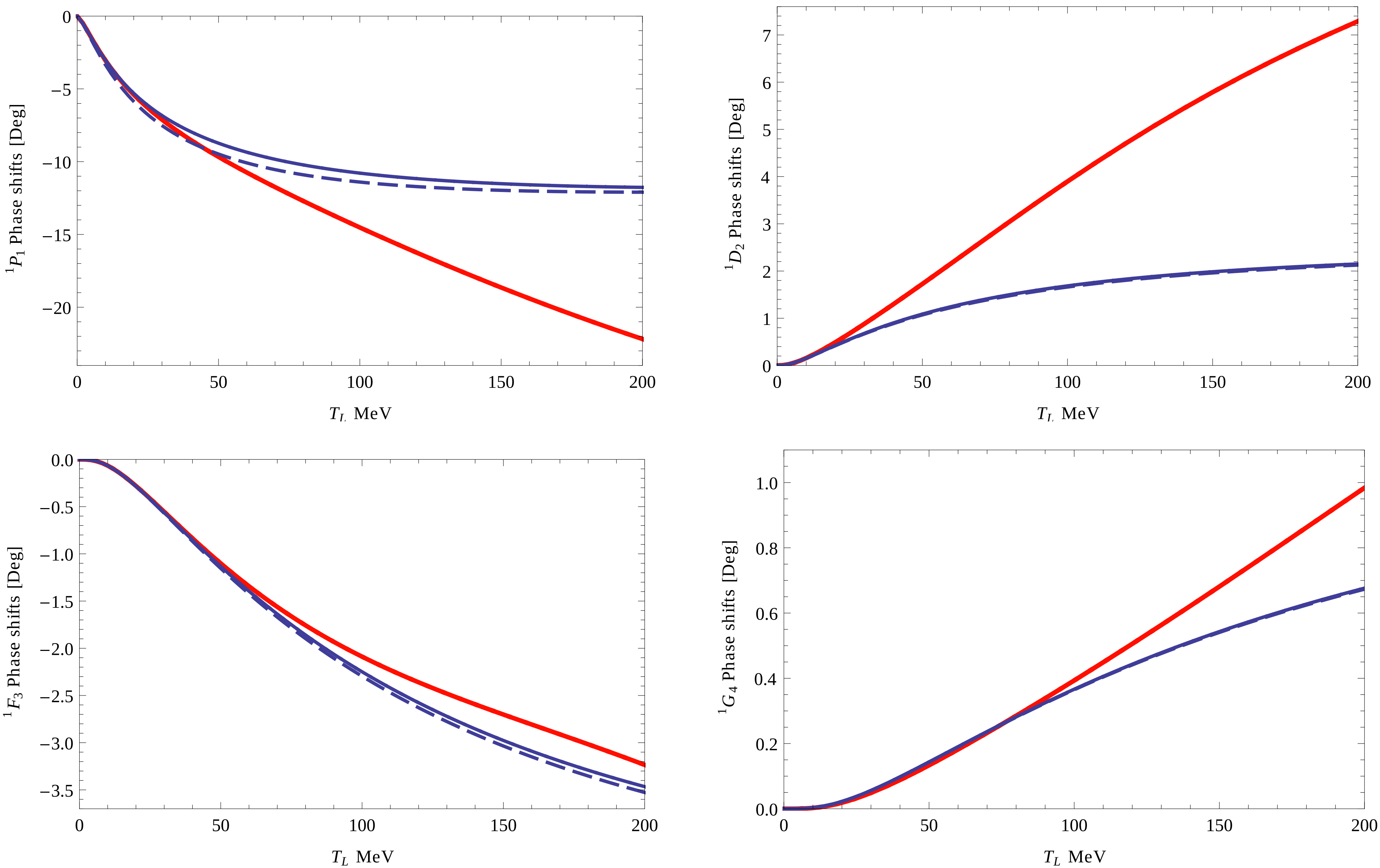
P. F. Bedaque, H. W. Hammer and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999)

we include a counter-term $\frac{c(\Lambda)\, p\, p'}{\Lambda^2}$ at LO.

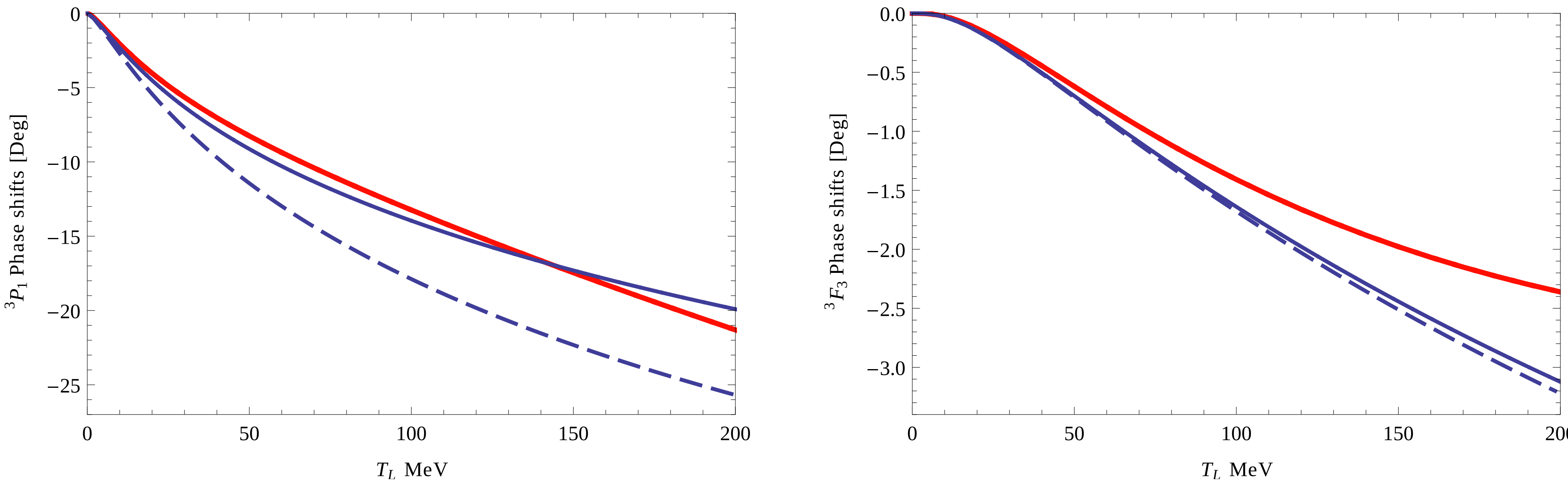
Due to its special dependence on Λ this counter-term does not upset the self-consistency of EFT.



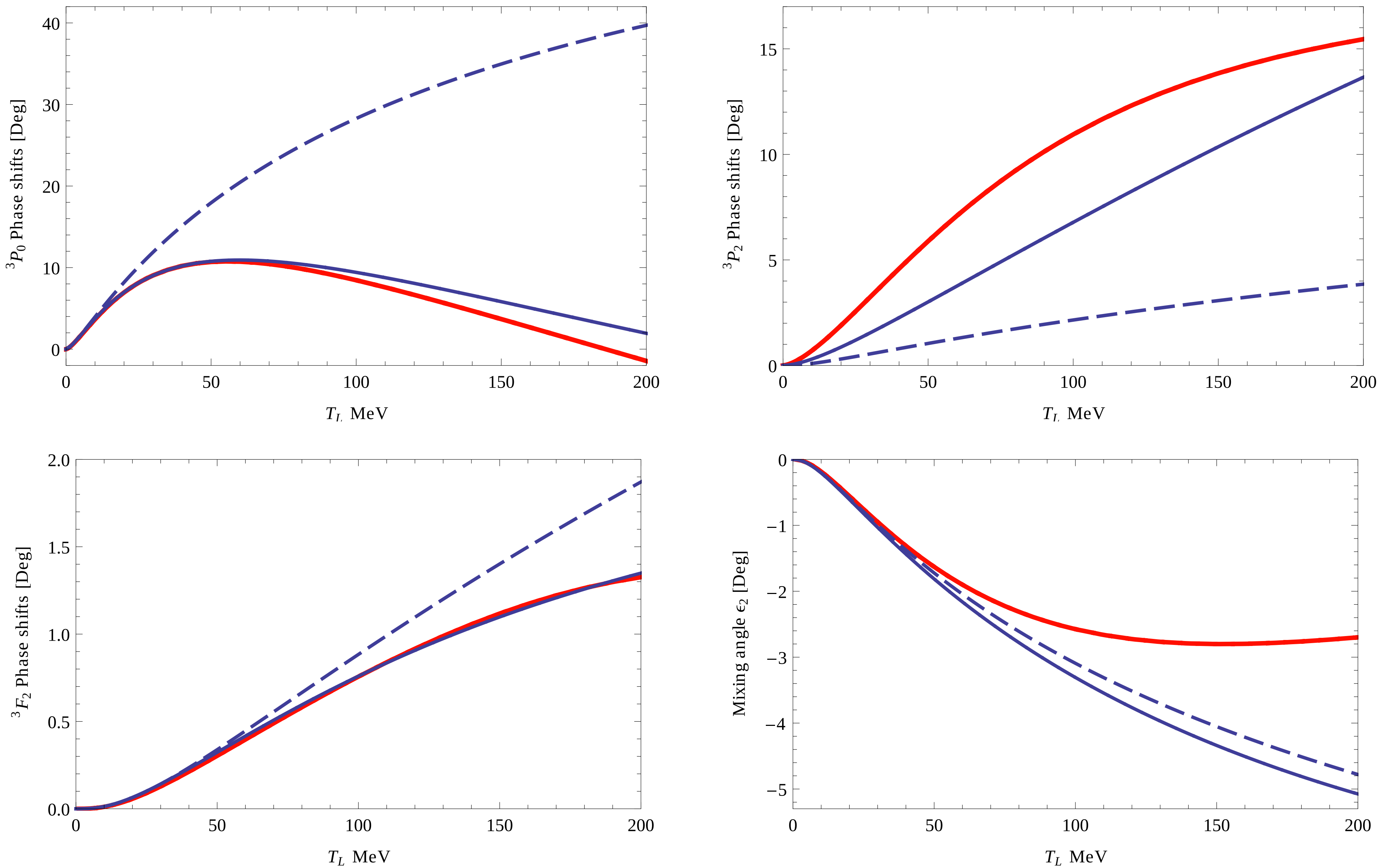
$^1S_0$ Phase shifts [Deg]
T_L MeV
$^3S_1$ Phase shifts [Deg]
T_L MeV
$^3D_1$ Phase shifts [Deg]
T_L MeV
Mixing angle ϵ_1 [Deg]
T_L MeV



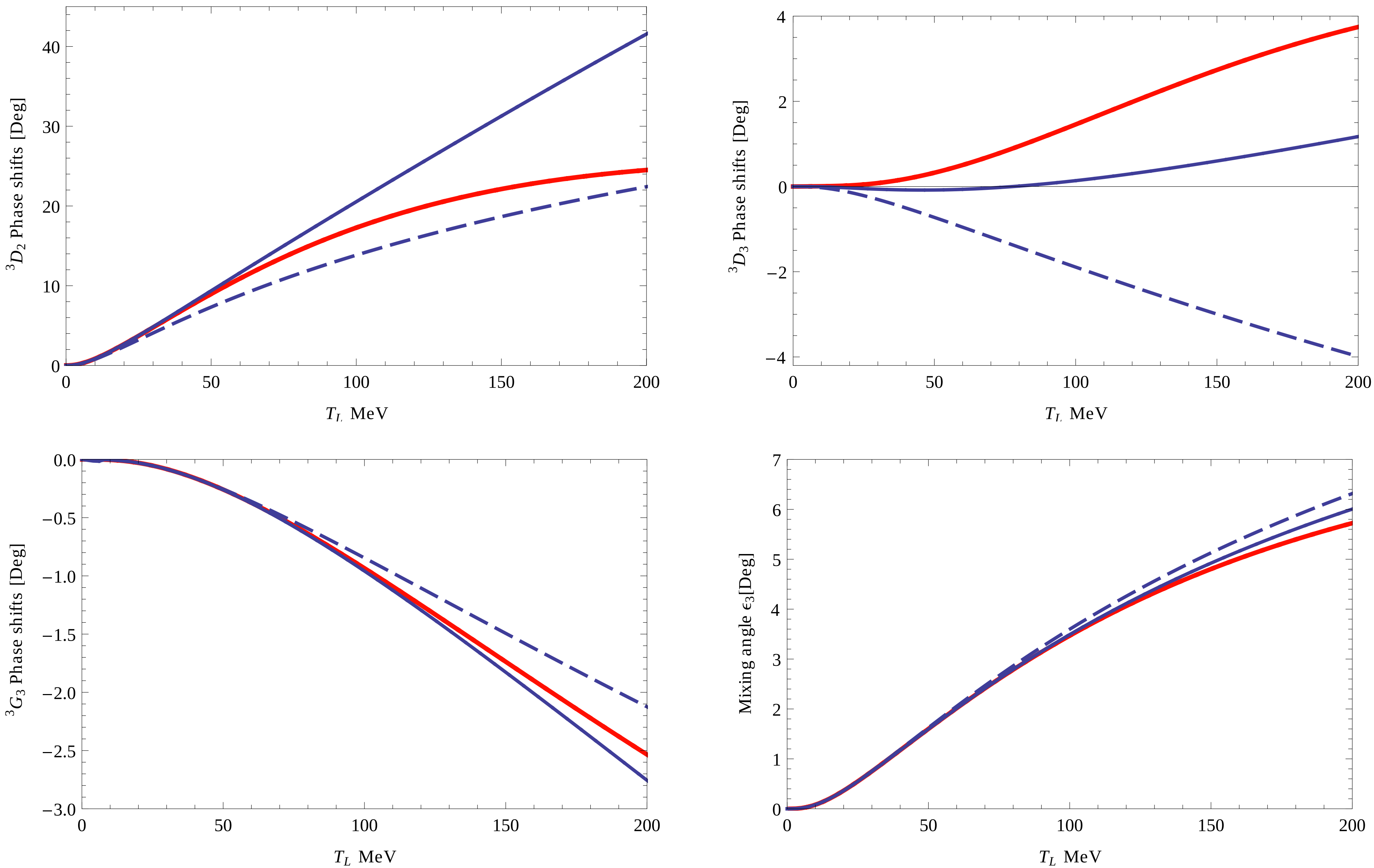
1P1 Phase shifts [Deg]
0
−5
−10
−15
−20
0
50
100
150
200
TL MeV
1D2 Phase shifts [Deg]
0
1
2
3
4
5
6
7
0
50
100
150
200
TL MeV
1F3 Phase shifts [Deg]
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
0
50
100
150
200
TL MeV
1G4 Phase shifts [Deg]
0.0
0.2
0.4
0.6
0.8
1.0
0
50
100
150
200
TL MeV



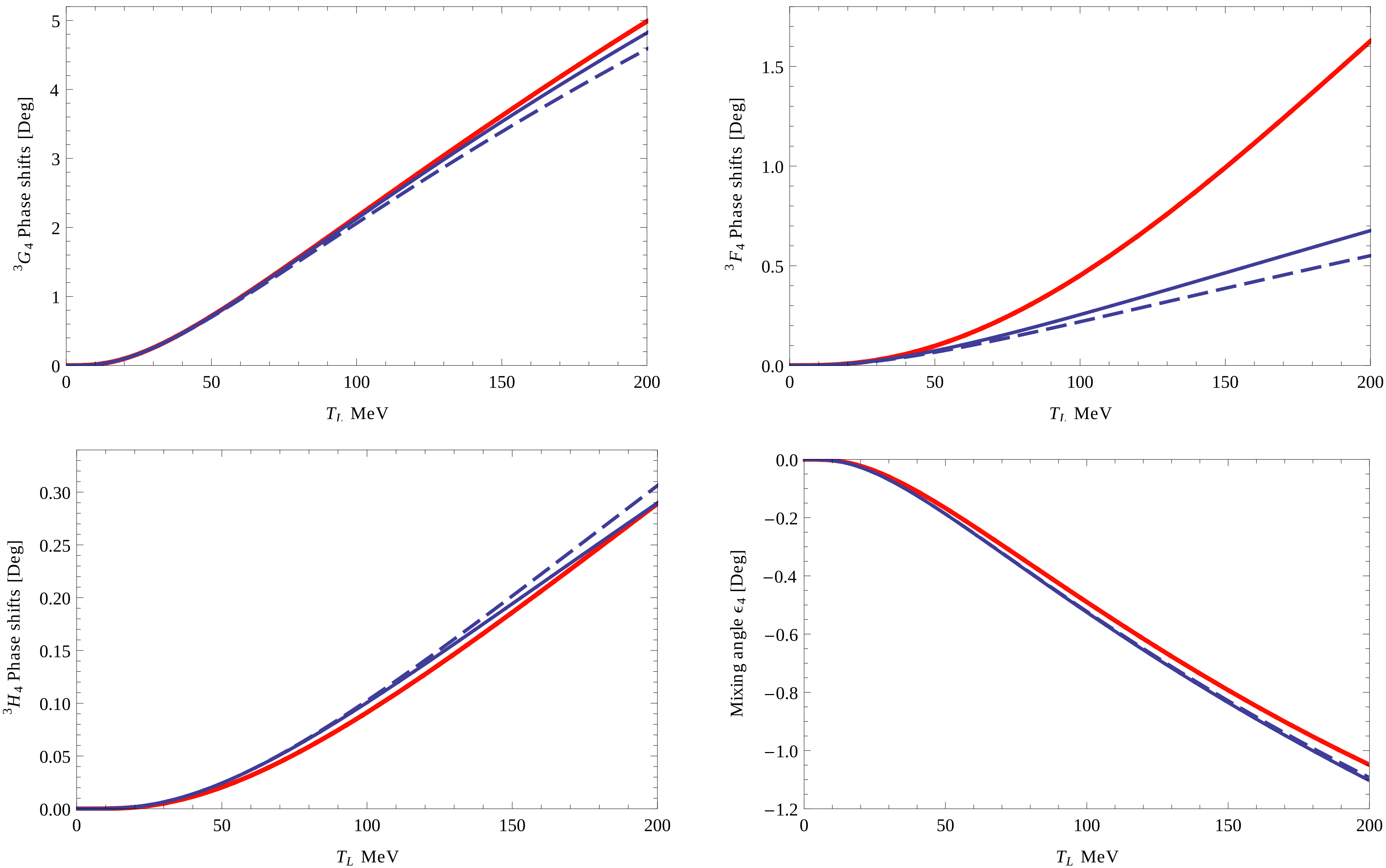
$^3P_1$ Phase shifts [Deg]
0
−5
−10
−15
−20
−25
0
50
100
150
200
T_L MeV
$^3F_3$ Phase shifts [Deg]
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
0
50
100
150
200
T_L MeV



$^3P_0$ Phase shifts [Deg]
T_L MeV
$^3P_2$ Phase shifts [Deg]
T_L MeV
$^3F_2$ Phase shifts [Deg]
T_L MeV
Mixing angle ϵ_2 [Deg]
T_L MeV



$^3D_2$ Phase shifts [Deg]
T_L MeV
$^3D_3$ Phase shifts [Deg]
T_L MeV
$^3G_3$ Phase shifts [Deg]
T_L MeV
Mixing angle ϵ_3[Deg]
T_L MeV



$^3G_4$ Phase shifts [Deg]
0
1
2
3
4
5
0
50
100
150
200
T_L MeV
$^3F_4$ Phase shifts [Deg]
0.0
0.5
1.0
1.5
0
50
100
150
200
T_L MeV
$^3H_4$ Phase shifts [Deg]
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0
50
100
150
200
T_L MeV
Mixing angle ϵ_4 [Deg]
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
0
50
100
150
200
T_L MeV

Summary

- We presented a modified version of Weinberg's approach to NN scattering problem.

- We use the Lorentz-invariant Lagrangian applying time-ordered perturbation theory.

- LO amplitude is obtained by solving an integral equation with contact interaction plus OPE.

- Corrections are calculated perturbatively.

- LO equation is perturbatively renormalizable.

- Effective OPE behaves as $\frac{1}{r^2}$ for $r \to 0$ in three space-time dimensions.

- In attractive partial waves OPE is screened by radial repulsion for non-vanishing orbital momenta.

- LO integral equation has unique solutions except for the $^3P_0$ channel.

- In $^3P_0$ channel we include a counter-term $\frac{c(\Lambda)\, p\, p'}{\Lambda^2}$ at LO.

- Due to the non-commutativity of the HB expansion with removed cutoff limit, it is not allowed to take the cutoff much larger than the nucleon mass within the HB formalism unless one includes the contributions of (an infinite number of) compensating terms of the effective Lagrangian.

Perturbative pions at next-to-leading order

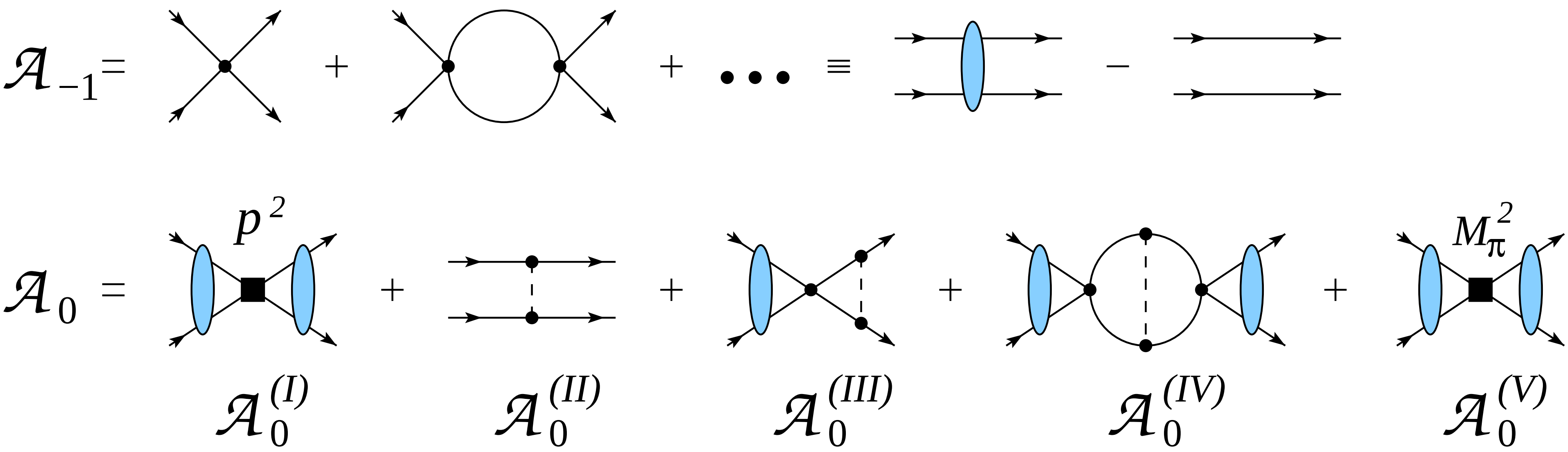
Here we adopt the normalization of the amplitude by KSW.

Scattering amplitude in the KSW approach

$$\mathcal{A} = \mathcal{A}_{-1} + \mathcal{A}_0 + \mathcal{A}_1 + \cdots ,$$

where the subscript indicates the power of the soft scale Q.

The LO contribution $\mathcal{A}_{-1}$ emerges from resummation of the LO contact interactions.


$\mathcal{A}_{-1} =$
$+$
$+ \cdots \equiv$
$-$
$\mathcal{A}_0 =$
p^2
M_π^2
$\mathcal{A}_0^{(I)}$
$\mathcal{A}_0^{(II)}$
$\mathcal{A}_0^{(III)}$
$\mathcal{A}_0^{(IV)}$
$\mathcal{A}_0^{(V)}$

The LO amplitude:

$$\mathcal{A}_{-1} = \frac{-C_0}{1 - C_0\, I(p)}\,.$$

The integral $I(p)$ is given by

$$\begin{aligned} I(p) &= \frac{m^2}{2}\frac{\mu^{3-n}}{(2\,\pi)^n}\int \frac{d^n k}{\left[k^2+m^2\right]\left[p_0 - \sqrt{k^2+m^2} + i\,0^+\right]} \\ &= \frac{1}{8\pi^2}[-(\bar\lambda + 2 - 2\ln\frac{m}{\mu})m^2 \\ &\quad - \frac{m^2}{\sqrt{m^2+p^2}}(\pi m + 2i\pi p - 2p\,\sinh^{-1}\left(\frac{p}{m}\right))]\,, \end{aligned}$$

with $\bar\lambda \equiv -1/(n-3) - \gamma - \ln(4\pi)$ and μ is the scale parameter of DR. C_0 is the linear combination of $C_{S,T}$. $p \equiv |\vec{p}\,|$, $k \equiv |\vec{k}\,|$.

Renormalization of $\mathcal{A}_{-1}$ is done by subtracting the loop integral at $p^2 = -\nu^2$ with $\nu \sim \mathcal{O}(Q)$,

$$I_{\mathsf{R}}(p,\nu) = I(p) - I(i\,\nu) = -\frac{m(\nu + i\,p)}{4\pi} + \mathcal{O}(p^2,\nu^2)\,,$$

and replacing C_0 by $C_0^{\mathsf{R}}(\nu)$:

$$\mathcal{A}_{-1} = \frac{-C_0^{\mathsf{R}}(\nu)}{1 - C_0^{\mathsf{R}}(\nu)\,I_{\mathsf{R}}(p,\nu)}\,.$$

The renormalized $\mathcal{A}_{-1}$ agrees with the KSW result modulo higher-order terms emerging from the $1/m$-expansion of $I_{\mathsf{R}}(p^2,\nu^2)$.

The renormalized contributions of the dressed subleading contact operators:

$$\mathcal{A}_0^{(I)} = \mathcal{A}_{-1}^2 \left[\frac{C_2^{\mathsf{R}}\, m^2 \left(2m^2 + p^2 - 2m\sqrt{m^2+p^2} \right)}{8\pi\, C_0^{\mathsf{R}}} - \frac{2\, C_2^{\mathsf{R}} p^2}{(C_0^{\mathsf{R}})^2} \right],$$

$$\mathcal{A}_0^{(V)} = -\frac{D_2^{\mathsf{R}} M_\pi^2}{(C_0^{\mathsf{R}})^2}\, \mathcal{A}_{-1}^2\,,$$

where $C_2^{\mathsf{R}} \equiv C_2^{\mathsf{R}}(\nu)$, $D_2^{\mathsf{R}} \equiv D_2^{\mathsf{R}}(\nu)$ are renormalized LECs.

$\mathcal{A}_0^{(V)}$ agrees with the HB result. HB result for $\mathcal{A}_0^{(I)}$ is entirely given by the second term in the brackets. For $C_0^{\mathsf{R}} \sim \mathcal{O}(Q^{-1})$ and $C_2^{\mathsf{R}} \sim \mathcal{O}(Q^{-2})$, the first term in the brackets is of order $\sim Q^3$ and the second one is of order $\sim Q^2$. Both approaches, therefore, again lead to the same result modulo corrections of a higher order.

Contributions involving pions:

Tree diagram yields the S-wave projected OPE potential,

$$\mathcal{A}_0^{(II)} = \frac{g_A^2}{4F_\pi^2}\left(-1 + \frac{M_\pi^2}{4\,p^2}\ln\frac{M_\pi^2+4p^2}{M_\pi^2}\right).$$

The renormalized contribution of the diagram with a pion line in one loop

$$\begin{aligned}
\mathcal{A}_0^{(III)} &= \frac{g_A^2}{2F_\pi^2}\mathcal{A}_{-1}\left[I_{\mathrm{R}}(p,\nu) - M_\pi^2 I_{1\mathrm{l}}(p)\right],\\
I_{1\mathrm{l}}(p) &= \frac{m^2}{2}\int\frac{d^nk}{(2\,\pi)^n}\frac{1}{\left[k^2+m^2\right]\left[p_0-\sqrt{k^2+m^2}\right]\left[(k-p)^2+M_\pi^2\right]}\\
&= -\frac{m}{8\pi p}\left[\tan^{-1}\left(\frac{2p}{M_\pi}\right) + \frac{i}{2}\ln\frac{M_\pi^2+4p^2}{M_\pi^2}\right] + \mathcal{O}\left(\frac{p}{m},\,\frac{M_\pi}{m}\right).
\end{aligned}$$

This agrees with the HB KSW result modulo higher order terms.

Contribution of the diagram with a pion line in two loops

$$\mathcal{A}_0^{(IV)} = \frac{g_A^2}{4F_\pi^2}\,\mathcal{A}_{-1}^2\left[M_\pi^2 I_{2l} - I_R(p,\nu)^2\right],$$

where the scalar two-loop integral has the form

$$I_{2l} \;=\; \frac{m^4}{4}\int \frac{d^nk_1 d^nk_2}{(2\,\pi)^{2\,n}} \frac{1}{\left[k_1^2+m^2\right]\left[p_0-\sqrt{k_1^2+m^2}+i\epsilon\right]\left[k_2^2+m^2\right]}$$
$$\times \frac{1}{\left[p_0-\sqrt{k_2^2+m^2}+i\epsilon\right]\left[(k_1-k_2)^2+M_\pi^2\right]}$$

$$\mathcal{A}_0^{(IV)} = \frac{g_A^2 m^2}{64\pi^2 F_\pi^2}\mathcal{A}_{-1}^2\Big\{M_\pi^2\Big[\frac{\ln 8}{4} - \frac{2G}{\pi} - \frac{7\zeta(3)}{2\pi^2} - \frac{1}{2}\ln\frac{M_\pi^2+4p^2}{m^2}$$
$$+ i\tan^{-1}\left(\frac{2p}{M_\pi}\right)\Big] - (\nu+ip)^2\Big\} + \cdots,$$

$G \approx 0.916$ is Catalan's constant and the ellipses refer to higher-order terms.

The corresponding HB result:

$$\mathcal{A}_{0,\,\mathrm{HB}}^{(IV)} = \frac{g_A^2 m^2}{64\pi^2 F_\pi^2}\mathcal{A}_{-1}^2\Big\{M_\pi^2\left[-\frac{1}{2}\ln\frac{M_\pi^2+4p^2}{\nu^2} + i\tan^{-1}\left(\frac{2p}{M_\pi}\right) + 1\right]$$
$$-(\nu+ip)^2\Big\}.$$

The difference in the constant terms in the square brackets can be compensated by a finite shift of the LEC D_2^{R}.

Non-polynomial terms in M_π^2 and p^2 are equal in both cases.

HB result has logarithmic dependence on ν which reflects the logarithmic divergence of $I_{2\text{loop}}$ when the integrand is approximated by the leading term in the $1/m$-expansion. It is, therefore, *necessary* to include $D_2 M_\pi^2$ in the HB approach at the same level as the diagrams, which appears at LO in the Weinberg approach.

In contrast, the original integral $I_{2\text{loop}}$ is finite and fulfills the power counting without any additional subtractions. Consequently there is no need to promote the $D_2 M_\pi^2$-term to LO within the modified Weinberg approach.