Transverse momentum-dependent parton distribution functions in lattice QCD

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“What is the probability of finding a quark with a given momentum k in a nucleon?”

Light cone coordinates: $w^{\pm} = \frac{1}{\sqrt{2}}(w^0 \pm w^3)$

Nucleon with large momentum along 3-axis: P^+ large, $P_T = 0$

Quark momentum components: $k^+ \sim P^+/m_N, \qquad k_T \sim 1, \qquad k^- \sim m_N/P^+$

Ask for distribution of quarks $f(x, k_T)$

- longitudinal momentum fraction $x = k^+/P^+$
- transverse momentum k_T

Definition of TMDs

Heuristically,

$$\Phi^{[\Gamma]}(x, k_T, P, S, \ldots) \equiv \text{“} \int dk^- \; \langle P, S| \; \bar{q}(k) \; \Gamma \; q(k) \; |P, S\rangle \; \Big|_{k^+ \equiv xP^+} \text{”}$$

Decompose in terms of TMDs, for example

$$\Phi^{[\gamma^+]}(x, k_T, P, S, \ldots) = f_1(x, k_T^2, \ldots) - \frac{\epsilon_{ij} k_i S_j}{m_N} f_{1T}^{\perp}(x, k_T^2, \ldots)$$

Definition of TMDs

More precisely, in terms of local operators,

$$\Phi^{[\Gamma]}(x, k_T, P, S, \ldots) \equiv \int \frac{d^2 b_T}{(2\pi)^2} \int \frac{d(b \cdot P)}{(2\pi) P^+} \exp\left(ix(b \cdot P) - i b_T \cdot k_T\right) \left. \frac{\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots)}{\widetilde{\mathcal{S}}(b^2, \ldots)} \right|_{b^+=0}$$

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S | \ \bar{q}(0) \ \Gamma \ \mathcal{U}[0, \ldots, b] \ q(b) \ | P, S \rangle$$

- “Soft factor” $\widetilde{\mathcal{S}}$ required to subtract divergences of Wilson line $\mathcal{U}$
- $\widetilde{\mathcal{S}}$ is typically a combination of vacuum expectation values of Wilson line structures
- Here, will consider only ratios in which soft factors cancel

Definition of TMDs

All leading twist structures:

$$\Phi^{[\gamma^+]} = f_1 - \left[\frac{\epsilon_{ij} k_i S_j}{m_N} f_{1T}^{\perp}\right] \text{odd}$$

$$\Phi^{[\gamma^+\gamma^5]} = \Lambda g_1 + \frac{k_T \cdot S_T}{m_N} g_{1T}$$

$$\Phi^{[i\sigma^{i+}\gamma^5]} = S_i h_1 + \frac{(2k_i k_j - k_T^2 \delta_{ij}) S_j}{2m_N^2} h_{1T}^{\perp} + \frac{\Lambda k_i}{m_N} h_{1L}^{\perp} + \left[\frac{\epsilon_{ij} k_j}{m_N} h_1^{\perp}\right] \text{odd}$$

Definition of TMDs

All leading twist structures:

$q \longrightarrow$ N $\downarrow$	U	L	T
U	f_1		$h_1^\perp$
L		g_1	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	h_1 $h_{1T}^\perp$

⟵ Boer-Mulders
(T-odd)

↑
Sivers (T-odd)

Fourier-transformed TMDs

$$\tilde{f}(x, b_T^2, \ldots) \equiv \int d^2k_T \, \exp(ib_T \cdot k_T) f(x, k_T^2, \ldots)$$

$$\tilde{f}^{(n)}(x, b_T^2, \ldots) \equiv n! \left(-\frac{2}{m_N^2} \partial_{b_T^2} \right)^n \tilde{f}(x, b_T^2, \ldots)$$

In limit $|b_T| \to 0$, recover k_T-moments:

$$\tilde{f}^{(n)}(x, 0, \ldots) \equiv \int d^2k_T \left(\frac{k_T^2}{2m_N^2} \right)^n f(x, k_T^2, \ldots) \equiv f^{(n)}(x)$$

ill-defined for large k_T, so will not attempt to extrapolate to $b_T = 0$, but give results at finite $|b_T|$.

Also, we can only access limited range of $b \cdot P$, so cannot Fourier-transform to obtain x-dependence. Therefore, consider only first x-moments (accessible at $b \cdot P = 0$):

$$f^{[1]}(k_T^2, \ldots) \equiv \int_{-1}^{1} dx \, f(x, k_T^2, \ldots)$$

Relation to physical processes

Context: All this is largely academic if we can't connect it to a physical measurement.

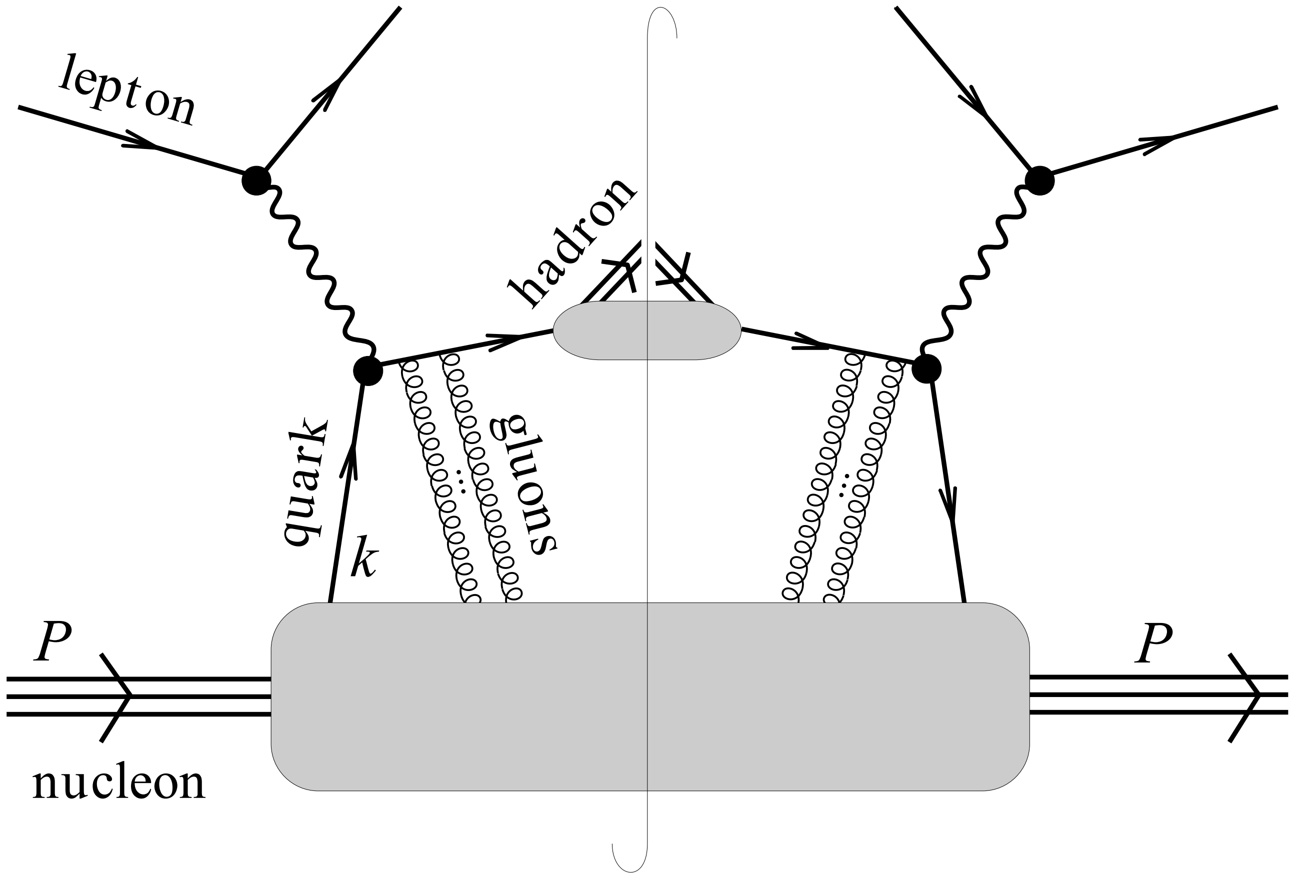
Factorization framework which allows one to separate cross section into hard amplitude, fragmentation function, TMD ?

In general, no! (e.g., processes with multiple hadrons in both initial and final state)

Factorization arguments have been given only for selected processes (SIDIS, Drell-Yan).

Physical process should also inform appropriate choice of gauge link $\mathcal{U}[0,\ldots,b]$

SIDIS (Semi-inclusive deep inelastic scattering)



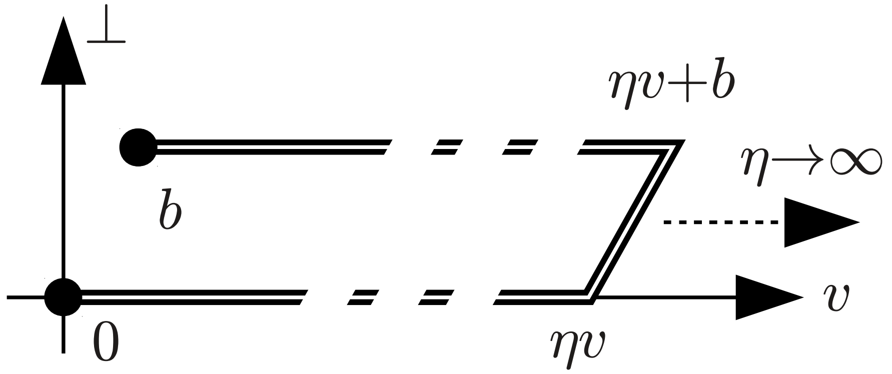
$l + N(P) \longrightarrow l' + h(P_h) + X$

Gauge link structure:

In matrix element $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \ldots) \equiv$

$\frac{1}{2}\langle P, S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \ldots, b]\ q(b)\ |P, S\rangle$

Staple-shaped gauge link $\mathcal{U}[0, \eta v, \eta v + b, b]$



incorporates SIDIS final state effects

Relation to physical processes

Staple-shaped links incorporate SIDIS final state effects:

- Gauge link roughly follows direction of ejected quark, (close to) light cone
- Effective resummed description of gluon exchanges between ejected quark and remainder of nucleon in evolving final state
- Beyond tree level: Rapidity divergences force taking staple direction slightly off the light cone. Approach of Aybat, Collins, Qiu, Rogers makes v space-like. Parametrize in terms of Collins-Soper parameter

  $$\hat{\zeta} \equiv \frac{P \cdot v}{|P||v|}$$

  Light-like staple for $\hat{\zeta} \to \infty$. Perturbative evolution equations for large $\hat{\zeta}$.
- In this approach, have "modified universality", $f^{\text{T-odd, SIDIS}} = -f^{\text{T-odd, DY}}$ (initial state interactions in DY case). SIDIS: $\eta v \cdot P \to \infty$, DY: $\eta v \cdot P \to -\infty$.

Without initial/final state effects, T-odd Sivers and Boer-Mulders functions would vanish!

Invariant amplitudes

Return to correlator

$$\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2}\langle P, S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ |P, S\rangle$$

Decompose in terms of invariant amplitudes; at leading twist,

$$\begin{aligned}
\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+]}_{\text{unsubtr.}} &= \widetilde{A}_{2B} + im_N\epsilon_{ij}b_iS_j\widetilde{A}_{12B} \\
\frac{1}{2P^+}\widetilde{\Phi}^{[\gamma^+\gamma^5]}_{\text{unsubtr.}} &= -\Lambda\widetilde{A}_{6B} + i[(b\cdot P)\Lambda - m_N(b_T\cdot S_T)]\widetilde{A}_{7B} \\
\frac{1}{2P^+}\widetilde{\Phi}^{[i\sigma^{i+}\gamma^5]}_{\text{unsubtr.}} &= im_N\epsilon_{ij}b_j\widetilde{A}_{4B} - S_i\widetilde{A}_{9B} \\
&\quad -im_N\Lambda b_i\widetilde{A}_{10B} + m_N[(b\cdot P)\Lambda - m_N(b_T\cdot S_T)]b_i\widetilde{A}_{11B}
\end{aligned}$$

Invariant amplitudes and TMDs

Conversely, invariant amplitudes directly give selected x-integrated TMDs in Fourier (b_T) space (showing just the ones relevant for Sivers, Boer-Mulders shifts):

$$\tilde{f}_1^{[1](0)}(b_T^2,\hat{\zeta},\ldots,\eta v\cdot P) = 2\widetilde{A}_{2B}(-b_T^2,0,\hat{\zeta},\eta v\cdot P)/\widetilde{S}(b^2,\ldots)$$

$$\tilde{f}_{1T}^{\perp[1](1)}(b_T^2,\hat{\zeta},\ldots,\eta v\cdot P) = -2\widetilde{A}_{12B}(-b_T^2,0,\hat{\zeta},\eta v\cdot P)/\widetilde{S}(b^2,\ldots)$$

$$\tilde{h}_1^{\perp[1](1)}(b_T^2,\hat{\zeta},\ldots,\eta v\cdot P) = 2\widetilde{A}_{4B}(-b_T^2,0,\hat{\zeta},\eta v\cdot P)/\widetilde{S}(b^2,\ldots)$$

Generalized shifts

Form ratios in which soft factors, (Γ-independent) multiplicative renormalization factors cancel

Sivers shift:

$$\langle k_y \rangle_{TU} \equiv m_N \frac{f_{1T}^{\perp [1](1)}}{f_1^{[1](0)}} = \frac{\int dx \int d^2k_T \, k_y \Phi^{[\gamma^+]}(x, k_T, S_T = (1,0))}{\int dx \int d^2k_T \, \Phi^{[\gamma^+]}(x, k_T, S_T = (1,0))}$$

Average transverse momentum of unpolarized ("U") quarks orthogonal to the transverse ("T") spin of nucleon; normalized to the number of valence quarks. "Dipole moment" in $b_T^2 = 0$ limit, "shift".

Issue: k_T-moments in this ratio singular; generalize to ratio of Fourier-transformed TMDs at *nonzero* b_T^2,

$$\langle k_y \rangle_{TU}(b_T^2, \ldots) \equiv m_N \frac{\tilde{f}_{1T}^{\perp [1](1)}(b_T^2, \ldots)}{\tilde{f}_1^{[1](0)}(b_T^2, \ldots)}$$

(remember singular $b_T \to 0$ limit corresponds to taking k_T-moment). "Generalized shift".

Generalized shifts from amplitudes

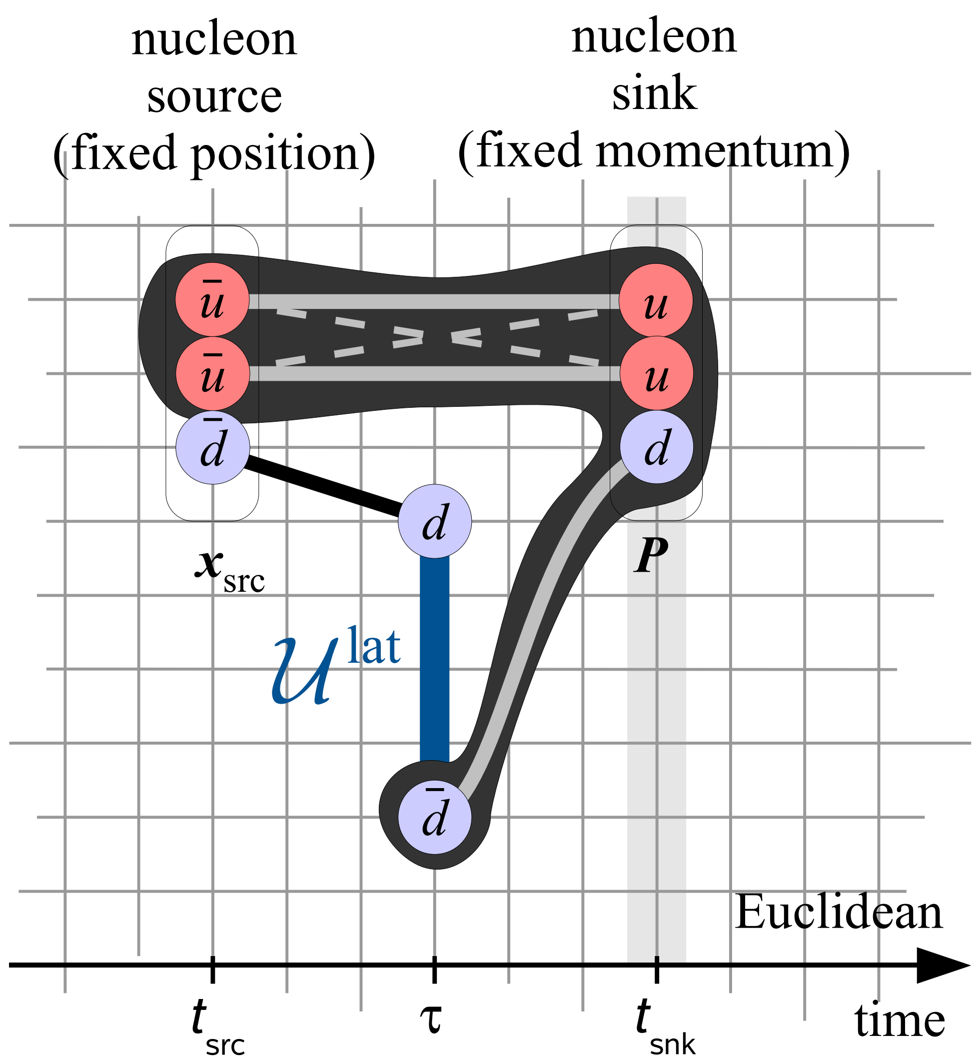
Now, can also express this in terms of invariant amplitudes:

$$\langle k_y \rangle_{TU}(b_T^2,\ldots) \equiv m_N \frac{\tilde{f}_{1T}^{\perp[1](1)}(b_T^2,\ldots)}{\tilde{f}_1^{[1](0)}(b_T^2,\ldots)} = -m_N \frac{\widetilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v\cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v\cdot P)}$$

Analogously, Boer-Mulders shift:

$$\langle k_y \rangle_{UT}(b_T^2,\ldots) = m_N \frac{\widetilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v\cdot P)}{\widetilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v\cdot P)}$$

Lattice setup



- Evaluate directly $\widetilde{\Phi}^{[\Gamma]}_{\text{unsubtr.}}(b, P, S, \hat{\zeta}, \mu)$

  $\equiv \frac{1}{2}\langle P,S|\ \bar{q}(0)\ \Gamma\ \mathcal{U}[0, \eta v, \eta v + b, b]\ q(b)\ |P,S\rangle$

- Euclidean time: Place entire operator at one time slice, i.e., b, ηv purely spatial
- Since generic b, v space-like, no obstacle to boosting system to such a frame!
- Parametrization of correlator in terms of $\widetilde{A}_i$ invariants permits direct translation of results back to original frame
- Form desired ratios of $\widetilde{A}_i$ invariants
- Extrapolate $\eta \to \infty$, $\hat{\zeta} \to \infty$ numerically

Lattice setup

Use three MILC 2+1-flavor gauge ensembles with $a \approx 0.12\,\mathrm{fm}$:

$m_\pi = 369\,\mathrm{MeV}\ ;\ 28^3 \times 64\ ;\ 2184$ samples

$m_\pi = 369\,\mathrm{MeV}\ ;\ 20^3 \times 64\ ;\ 5264$ samples

$m_\pi = 518\,\mathrm{MeV}\ ;\ 20^3 \times 64\ ;\ 3888$ samples

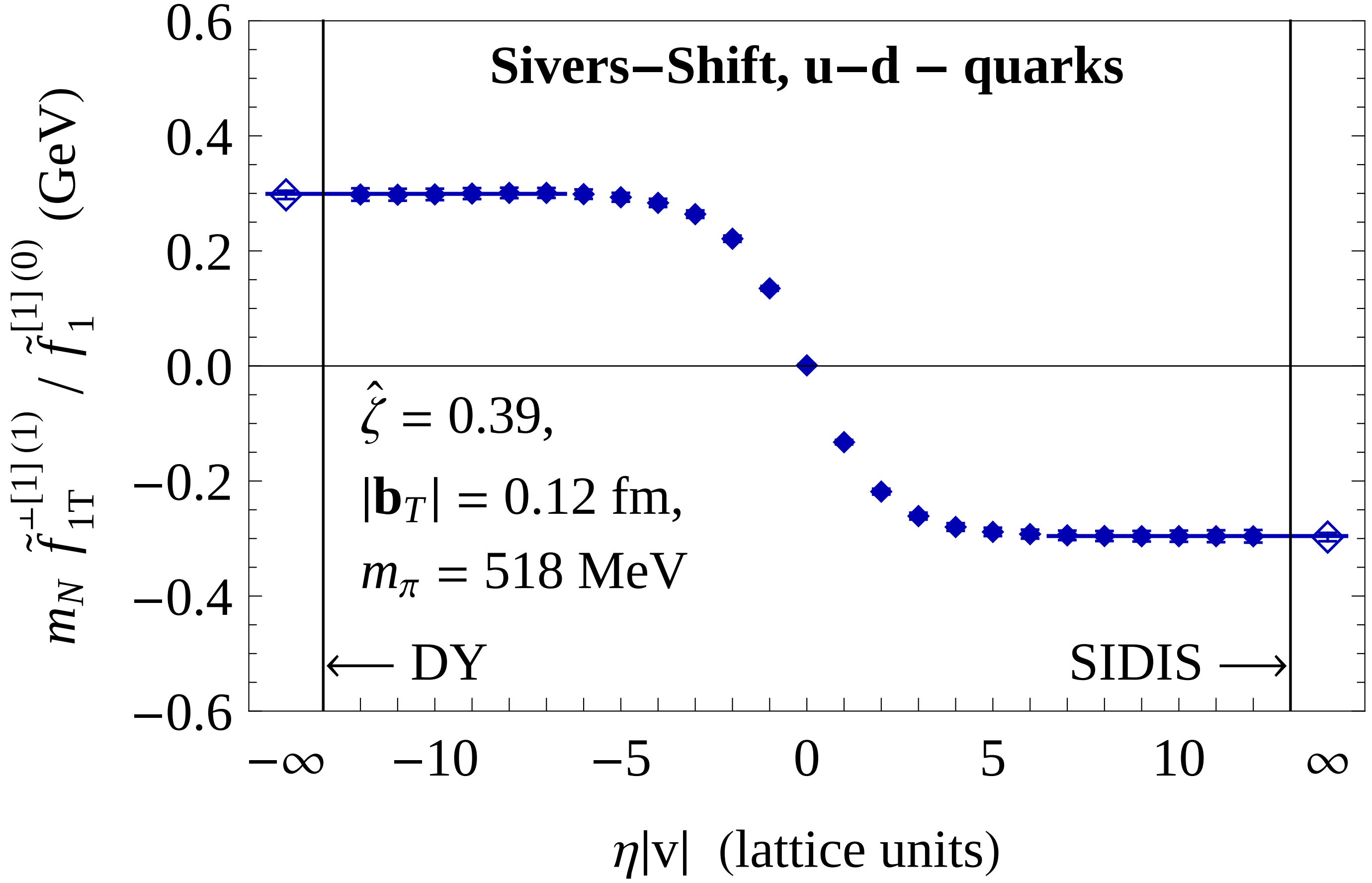
Sink momenta P: $(0,0,0)$, $(-1,0,0)$, $(-2,0,0)$, $(1,-1,0)$

Variety of b, ηv; note $b \perp P$, $b \perp v$ (lowest x-moment, kinematical choices/constraints)

Largest $\hat{\zeta} = 0.78$

Results: Sivers shift

Dependence on staple extent; sequence of panels at different $|b_T|$



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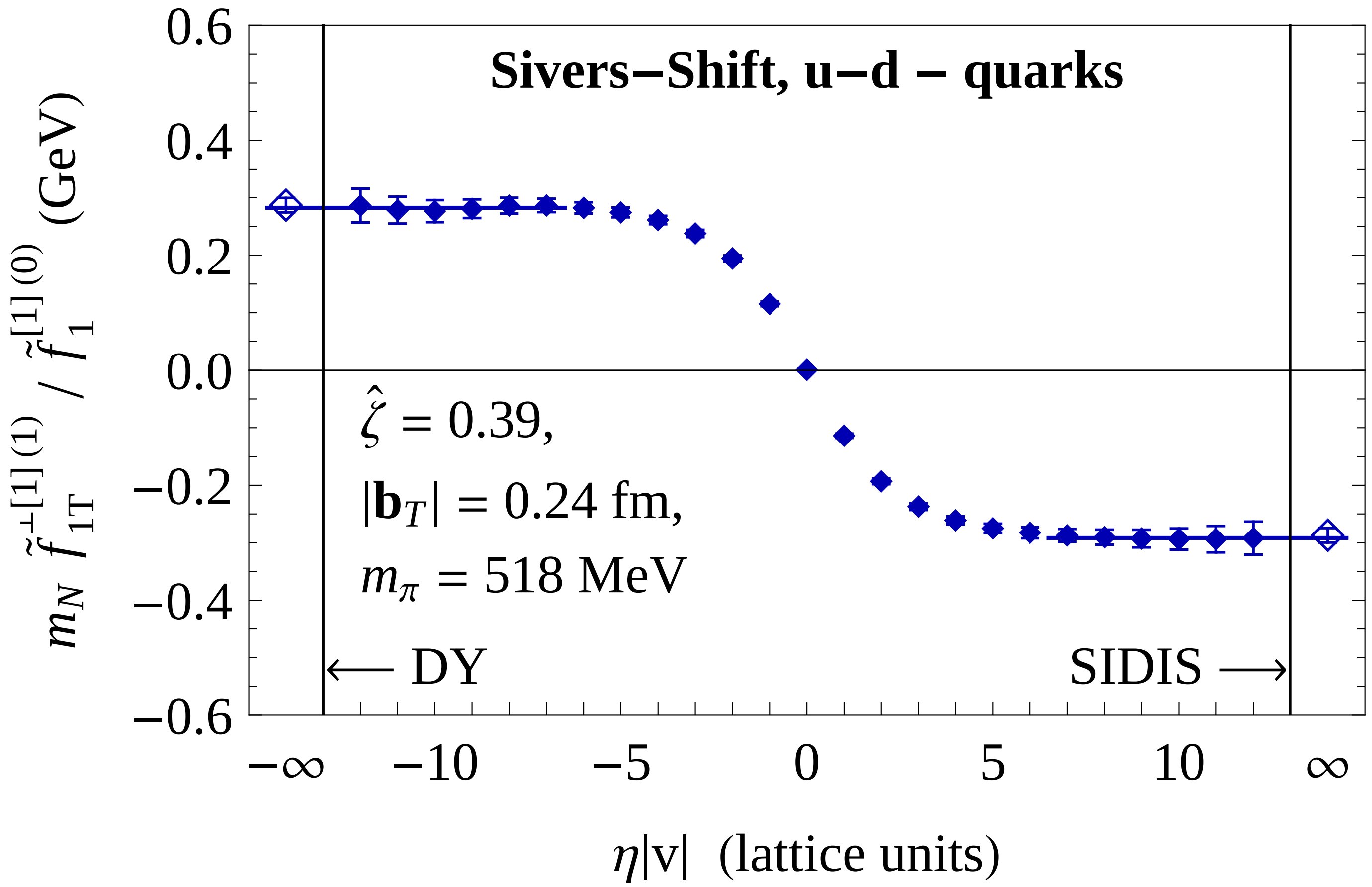

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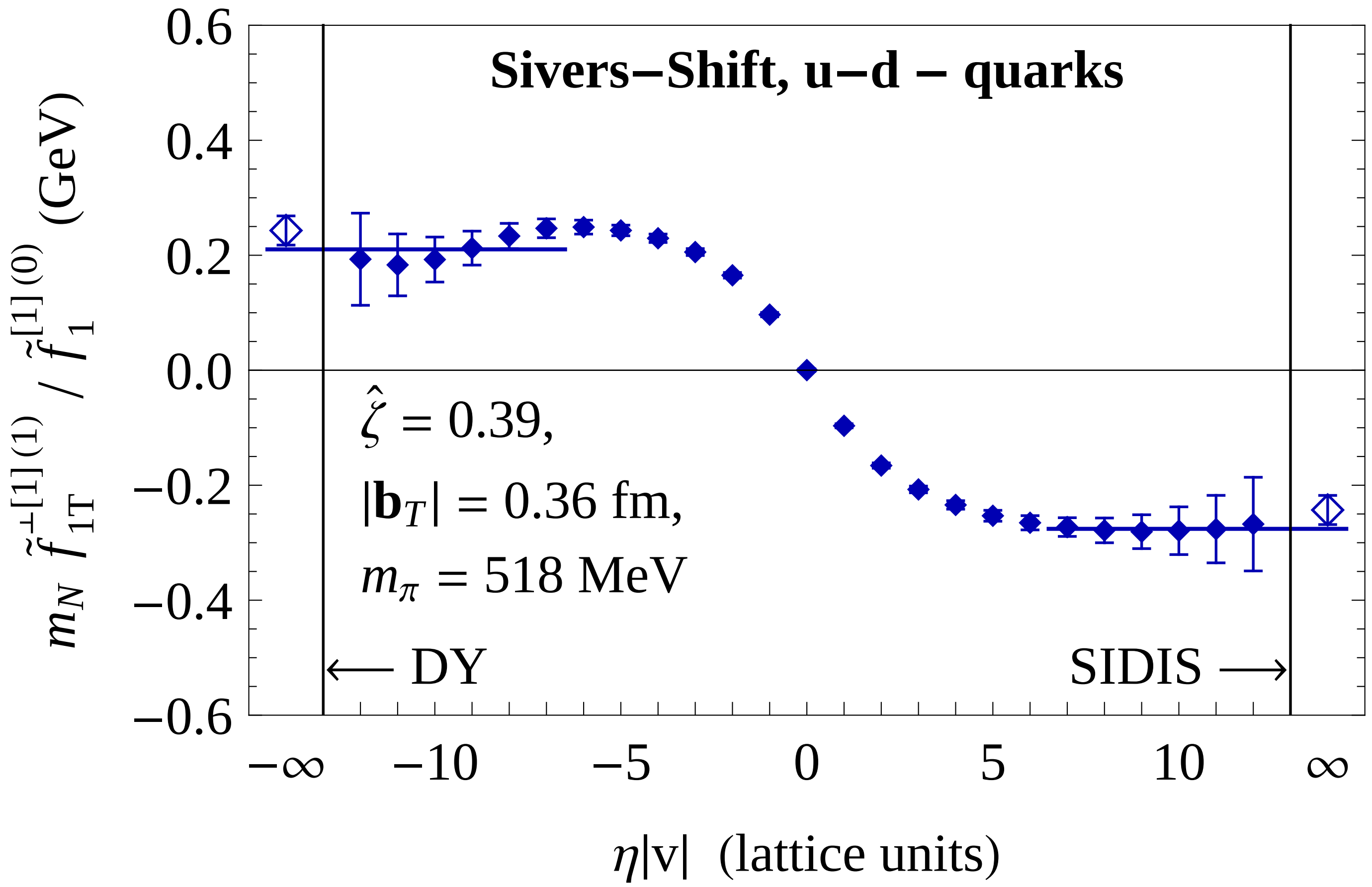

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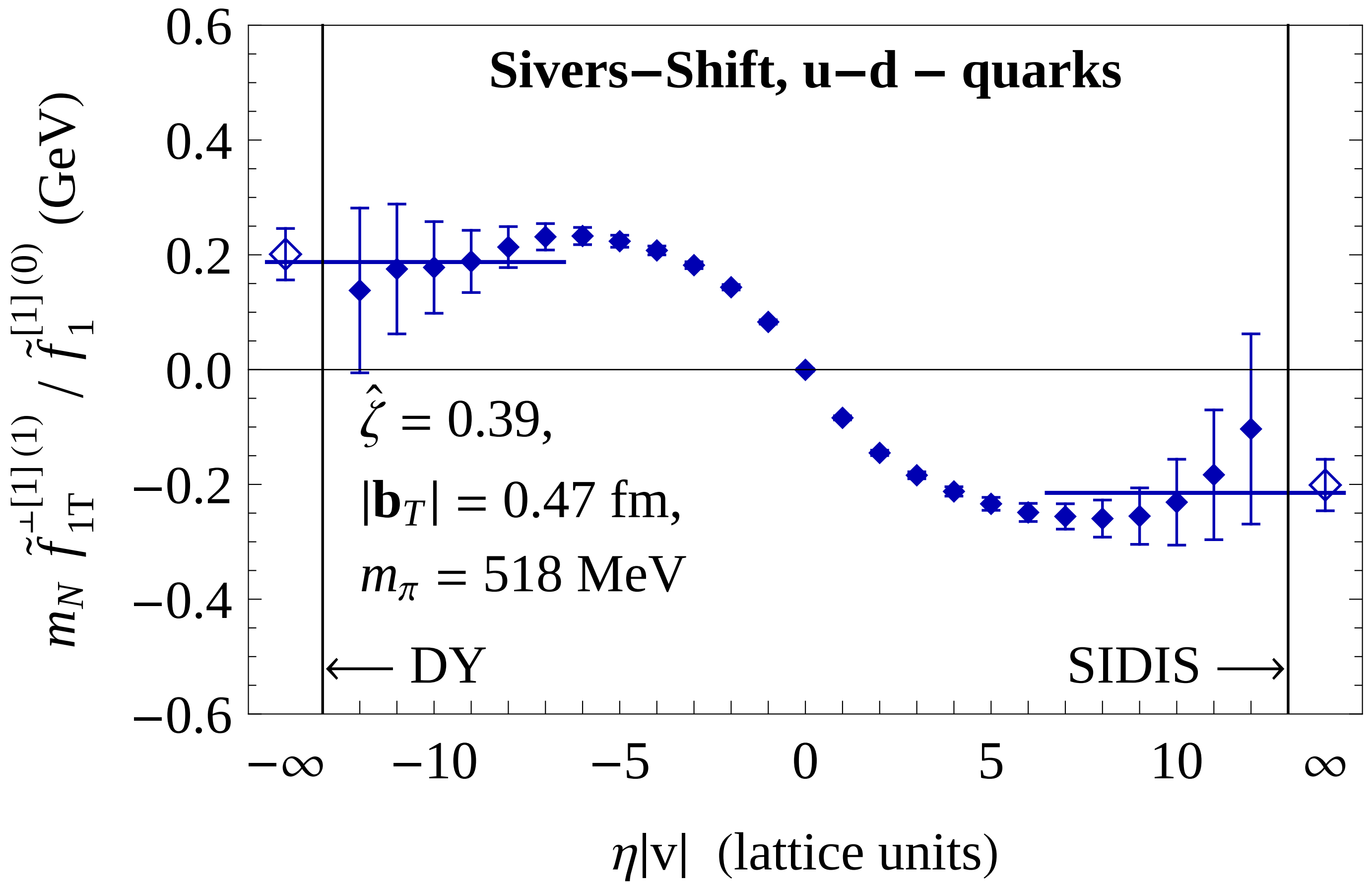

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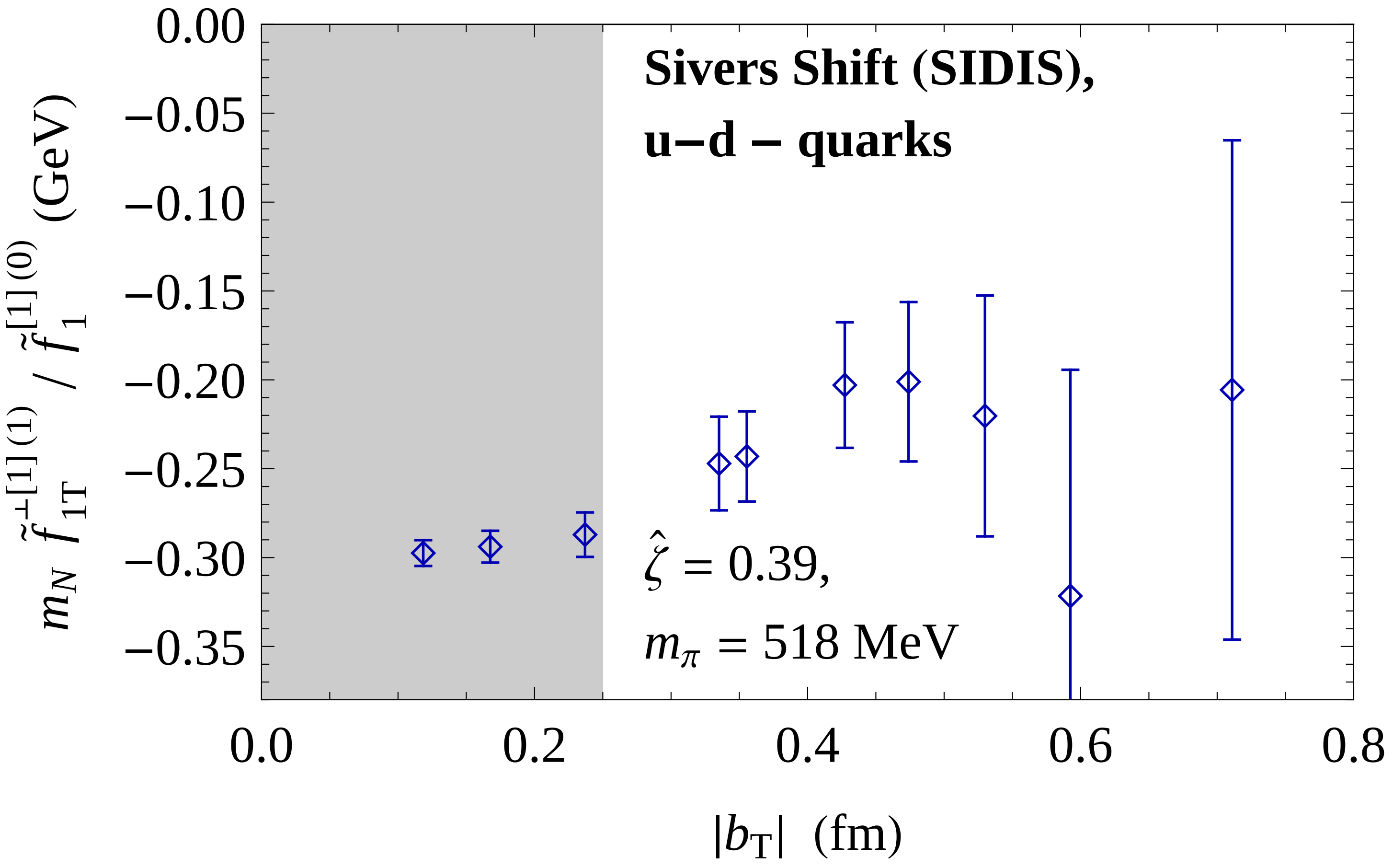
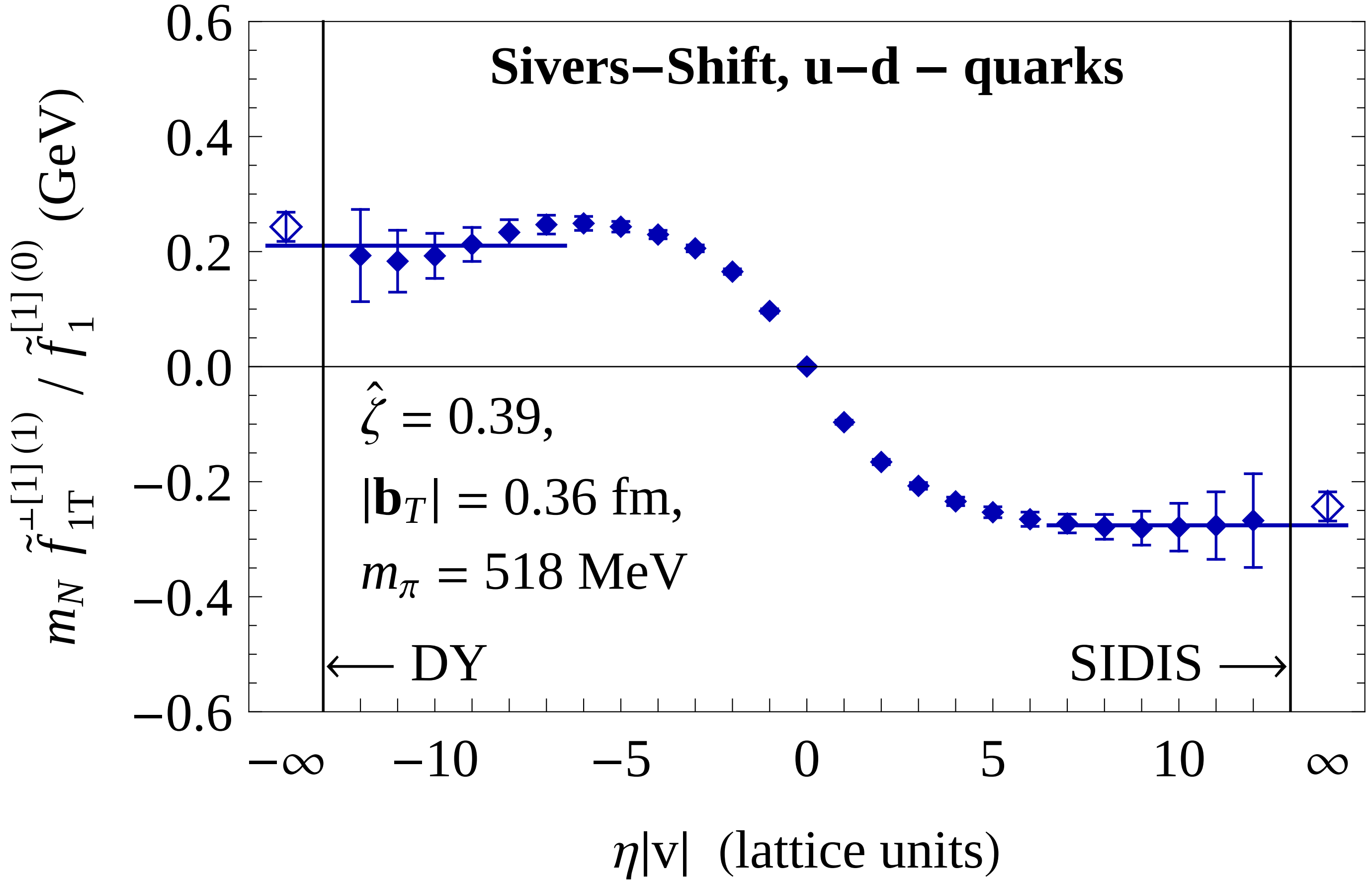
Dependence of SIDIS limit on $|b_T|$



Results: Sivers shift

Dependence on staple extent; sequence of panels at different $\hat{\zeta}$



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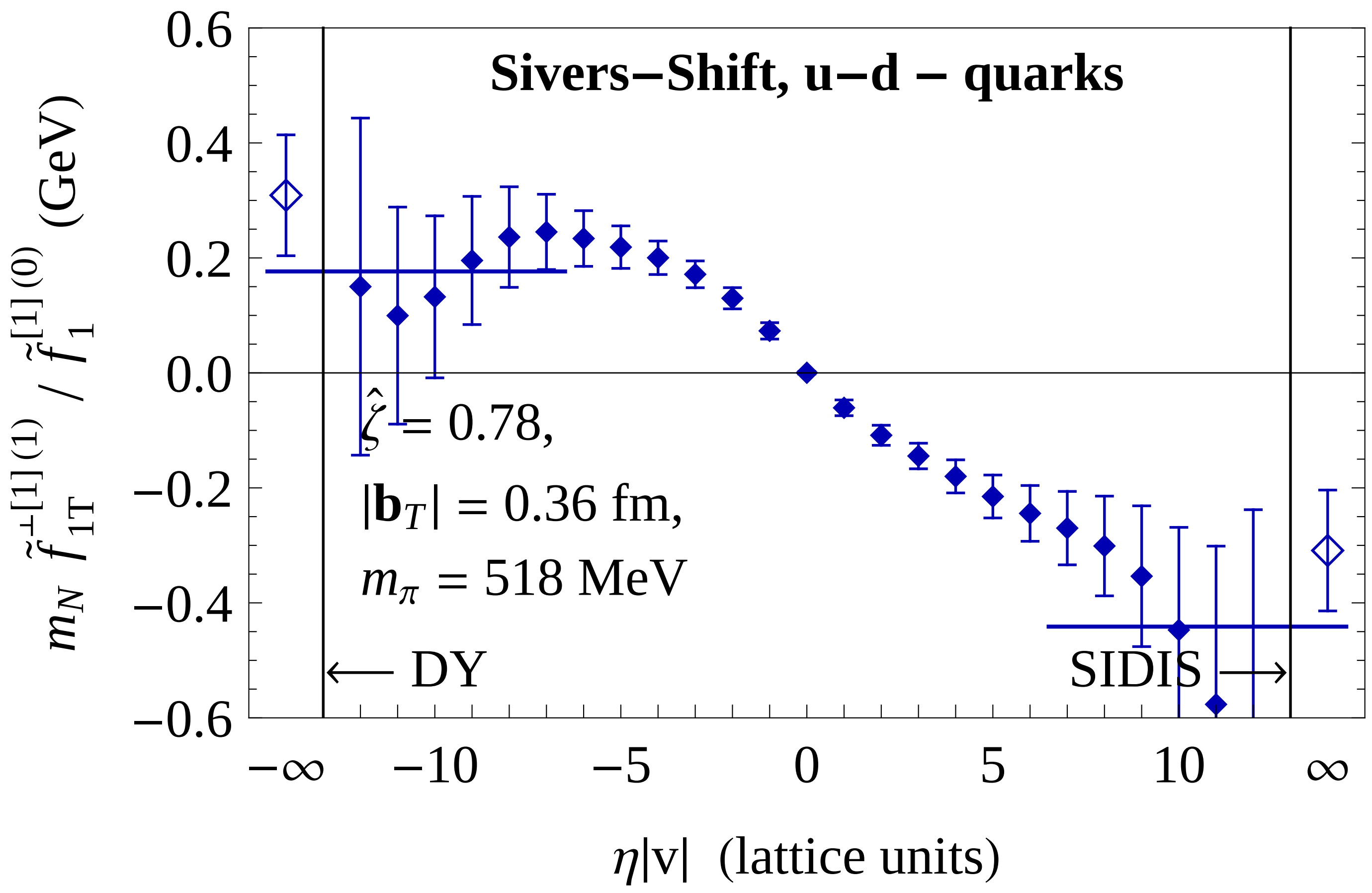

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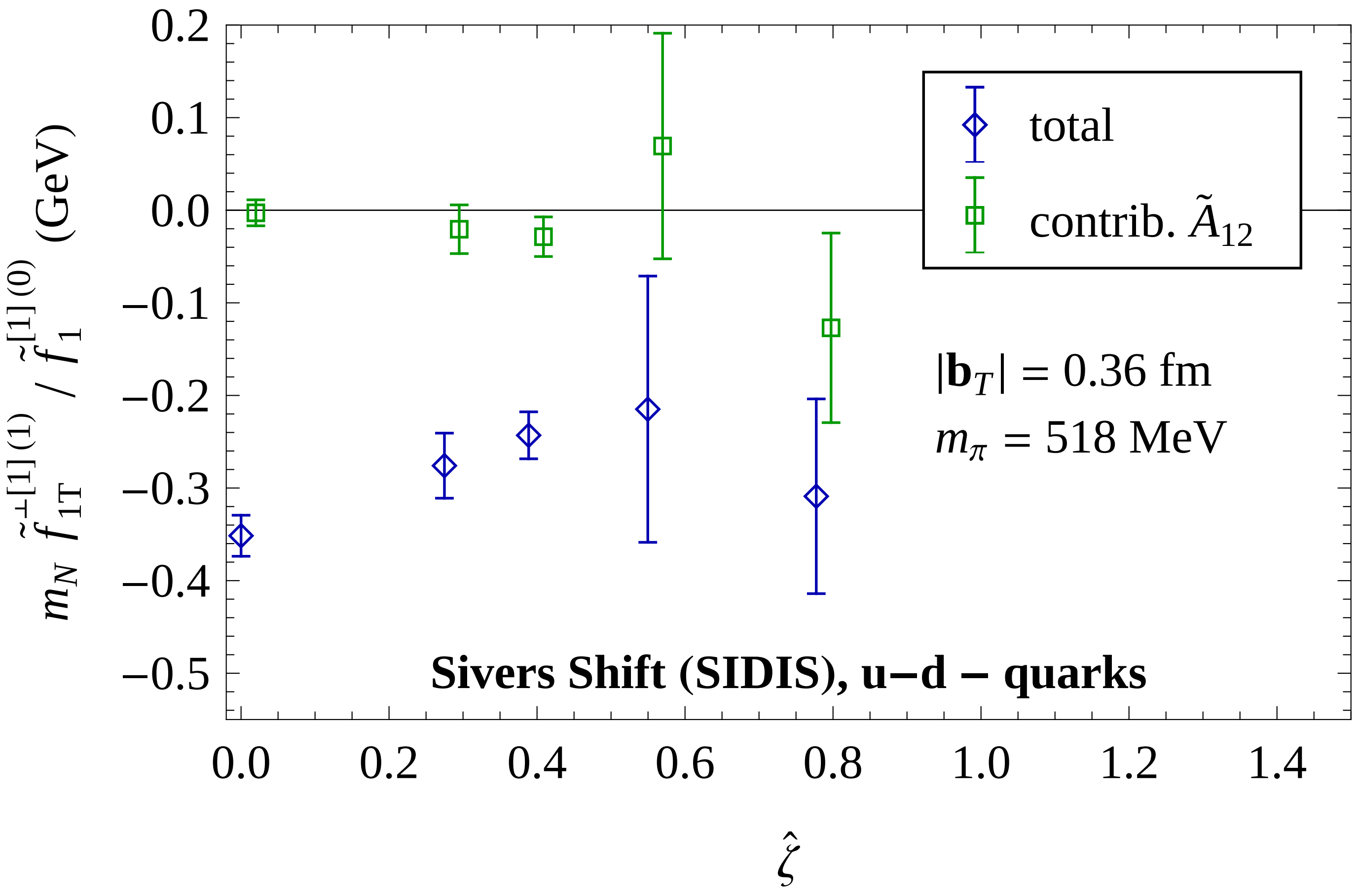
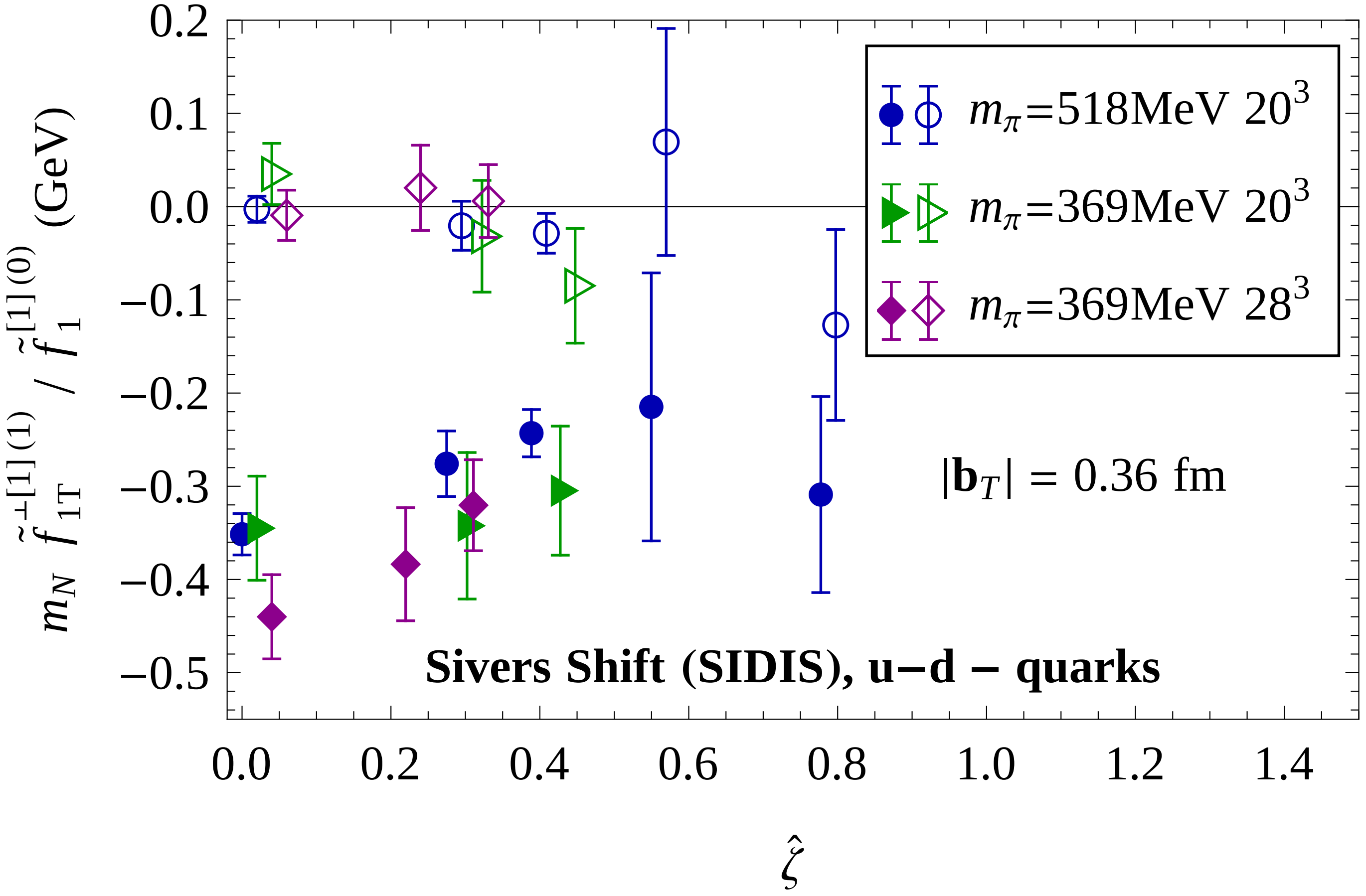


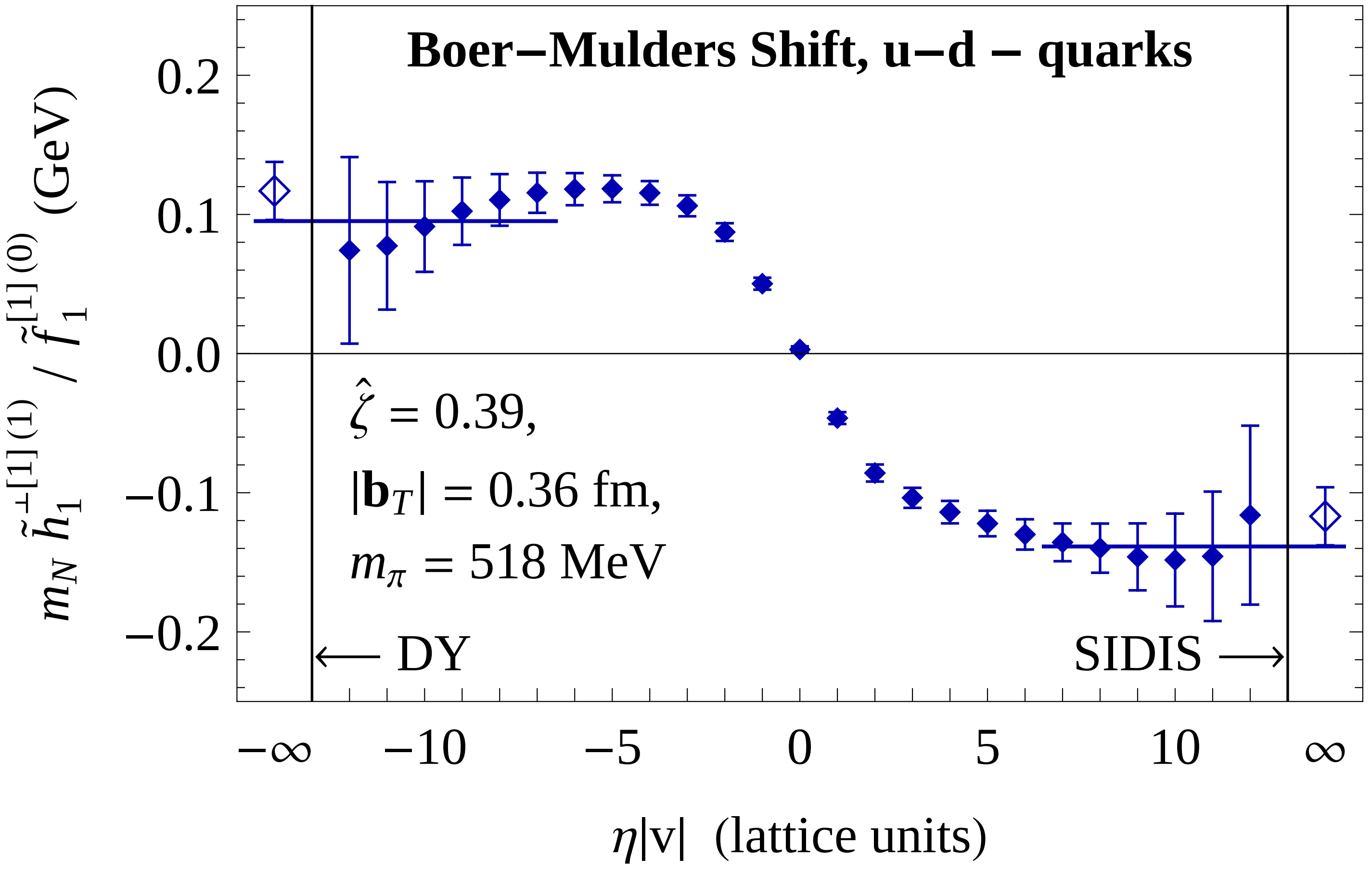
Results: Sivers shift

Dependence of SIDIS limit on $\hat{\zeta}$, all three ensembles



Results: Boer-Mulders shift

Dependence on staple extent



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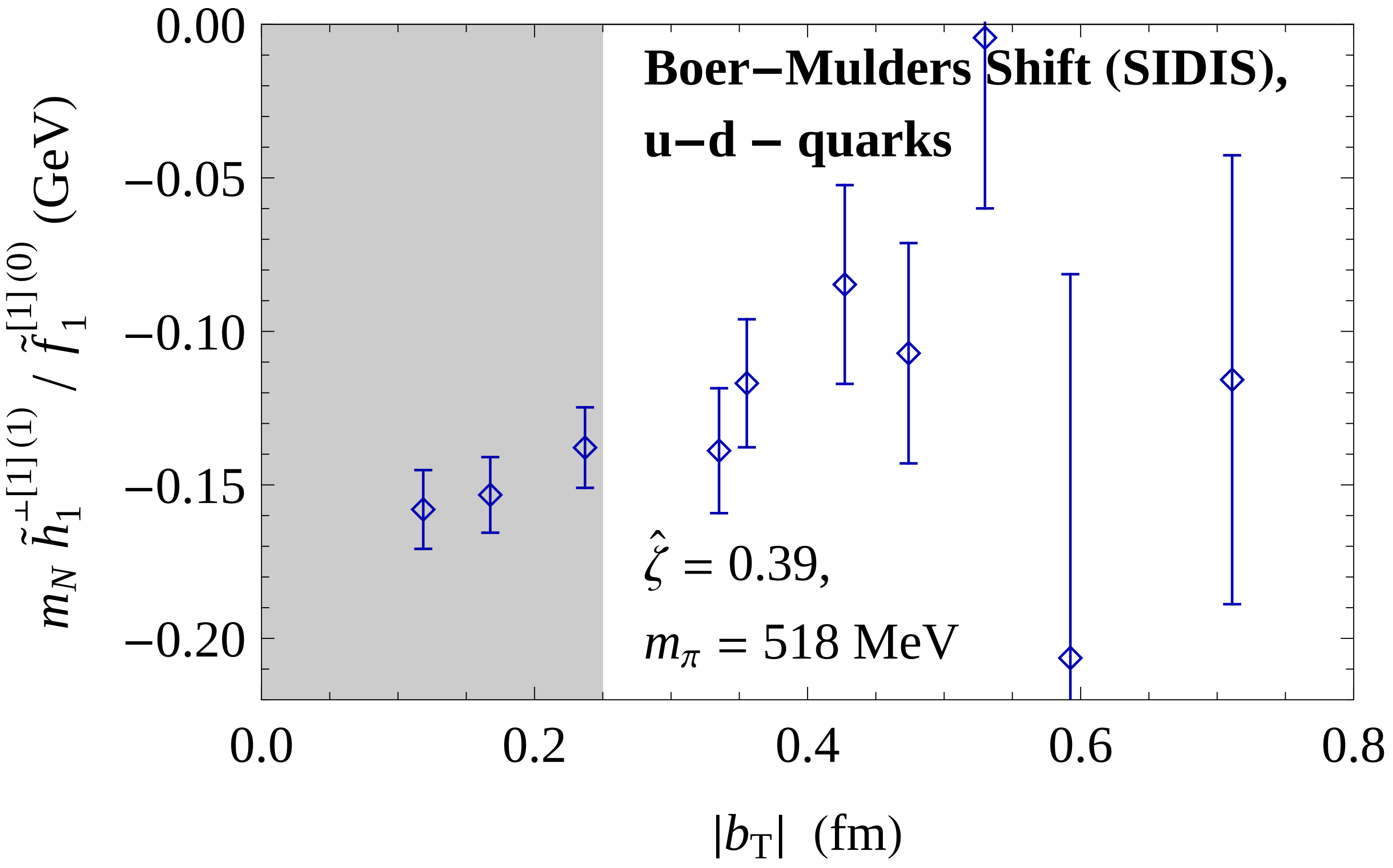
Dependence of SIDIS limit on $|b_T|$



Results: Boer-Mulders shift

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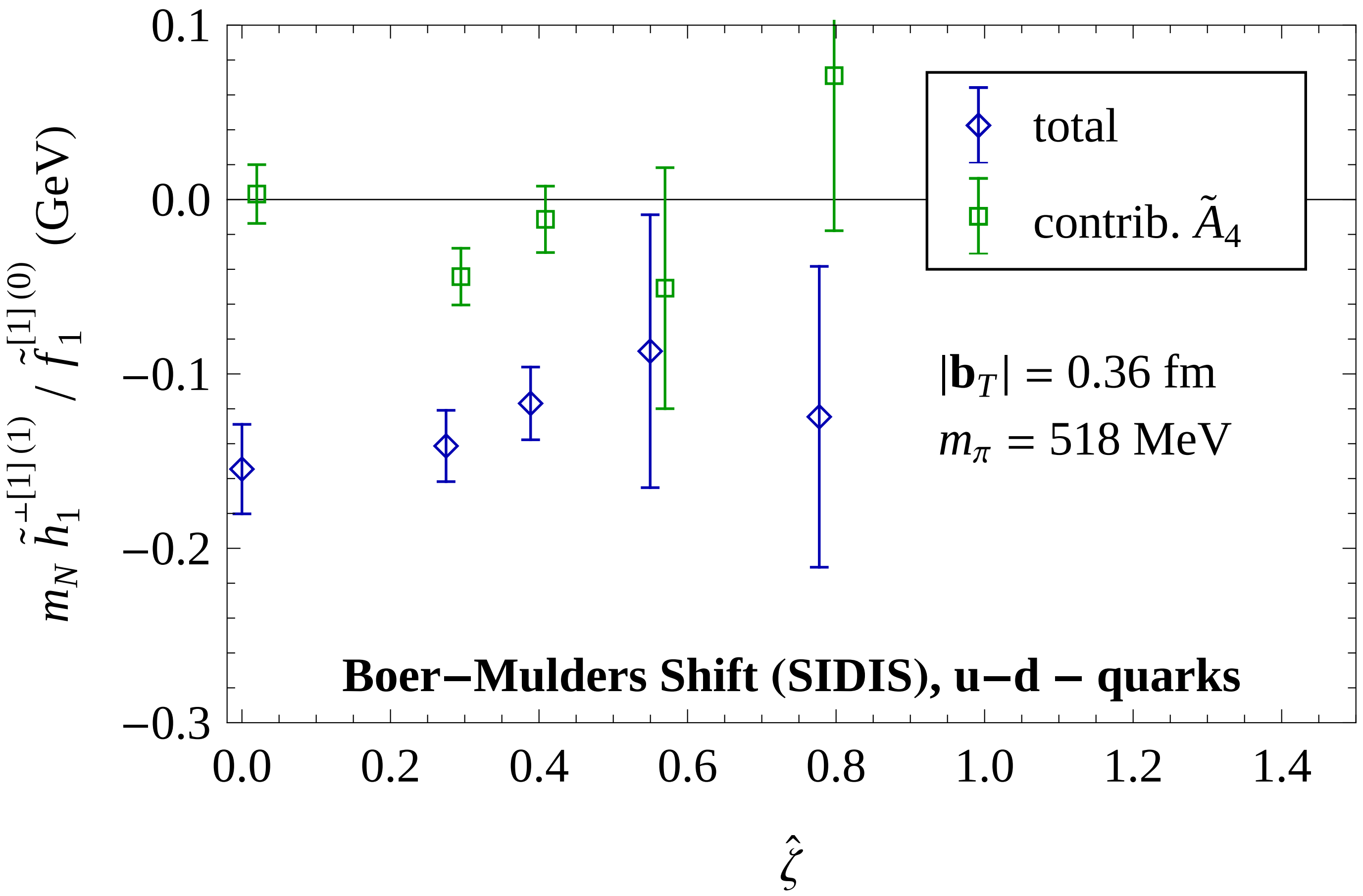

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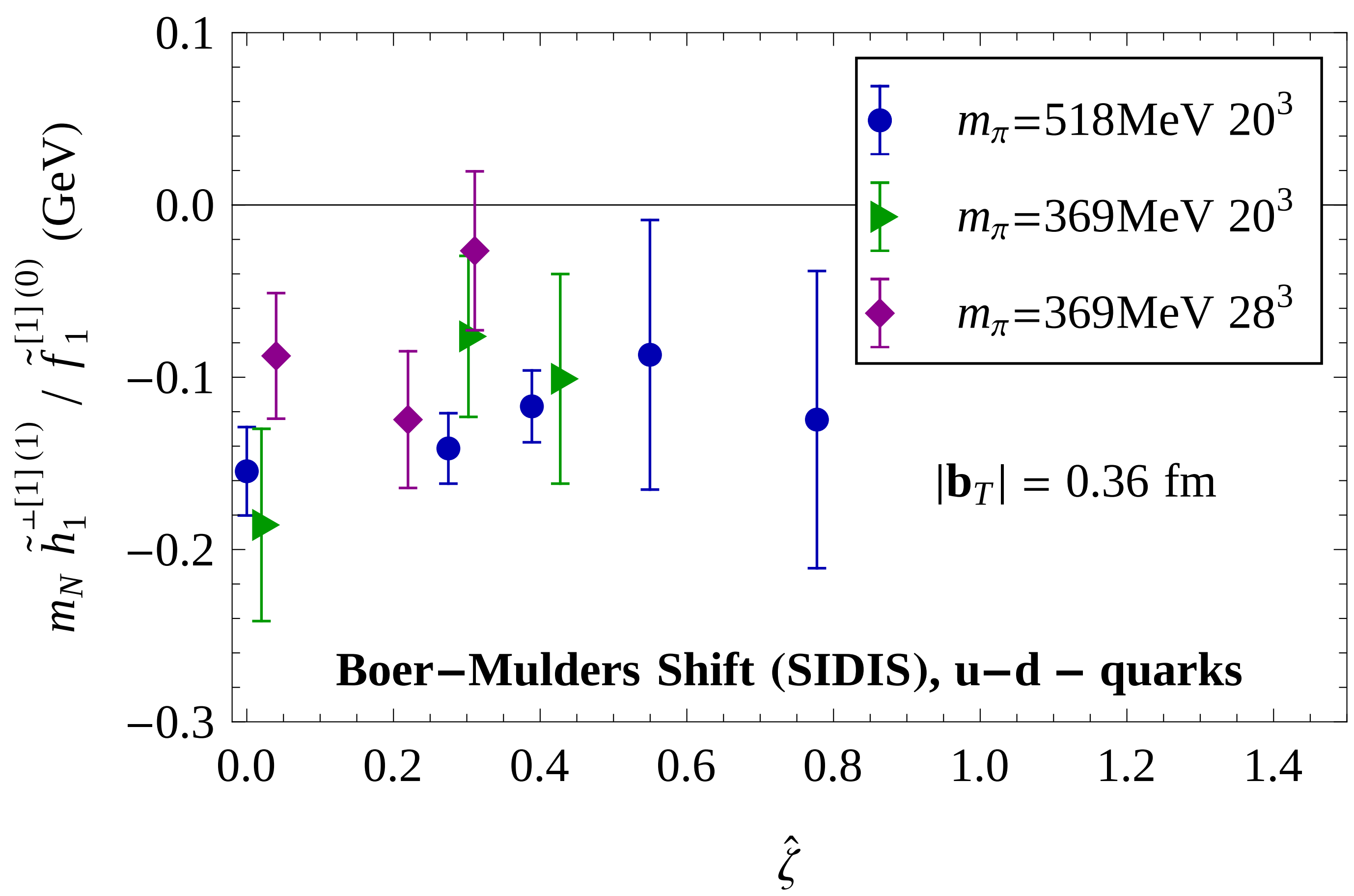
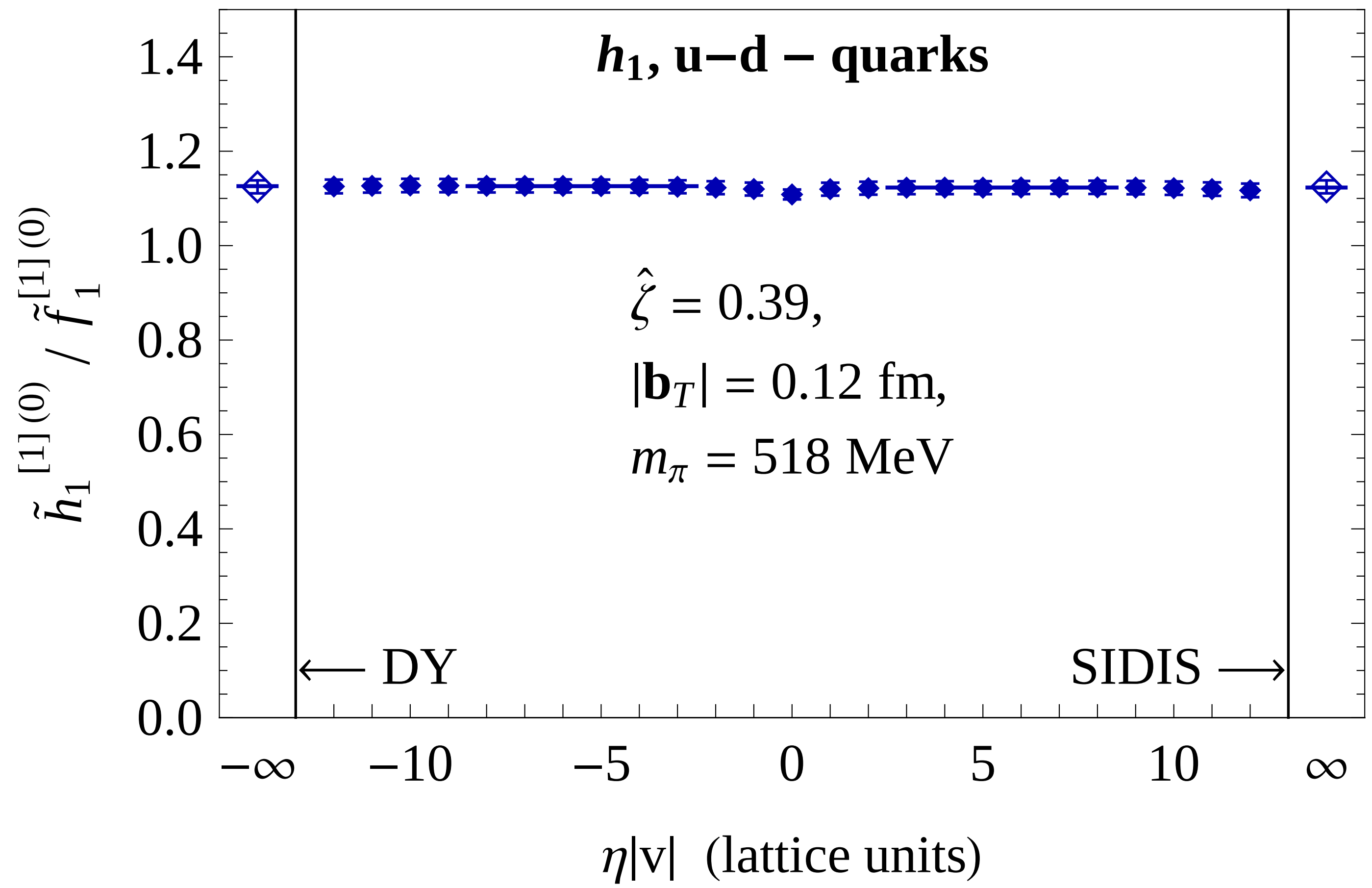


Conclusions

- Exploratory study of TMDs using staple-shaped gauge link structures
- Accessed T-odd Sivers, Boer-Mulders observables; SIDIS, DY limits distinguished by sign of $v \cdot P$. For u-d quark combination, SIDIS Sivers and Boer-Mulders TMDs both sizeable and negative.
- To avoid soft factors, multiplicative renormalization constants, constructed appropriate ratios of Fourier-transformed TMDs (“shifts”).
- v taken off light cone: Dependence on Collins-Soper parameter $\hat{\zeta}$. In addition to $\eta v \rightarrow \infty$, need to also consider $\hat{\zeta} \rightarrow \infty$.
- $\eta v \rightarrow \infty$ seems under good control; plateaux reached at moderate values.
- $\hat{\zeta} \rightarrow \infty$ remains a challenge. No clear trends seen in the data sets available. Need much larger $\hat{\zeta}$. Presently investigating pion with this in mind.
- No significant volume dependence, pion mass dependence detected within the limited set of (three) cases considered

Results: Transversity

Dependence on staple extent; sequence of panels at different $|b_T|$



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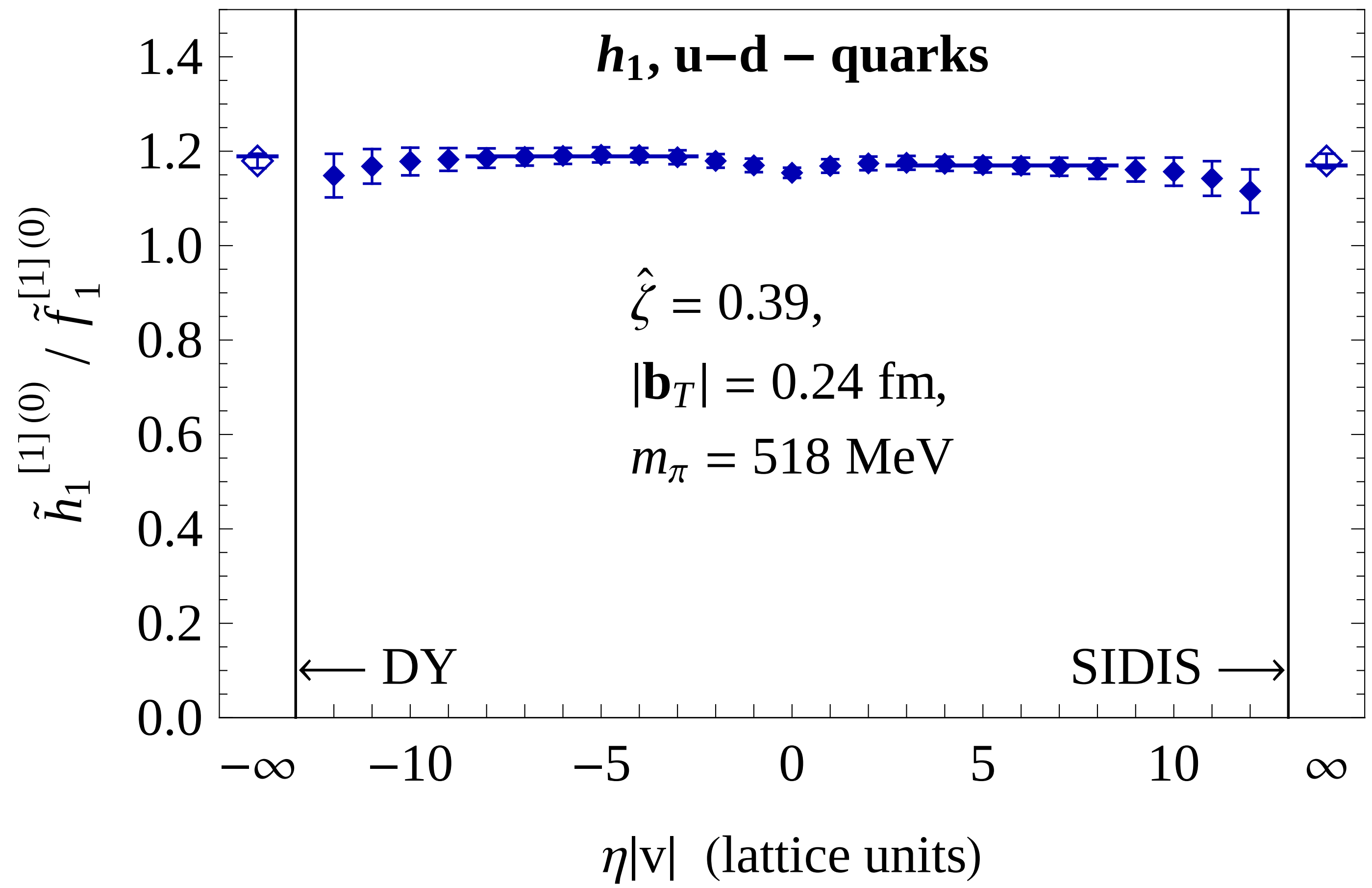

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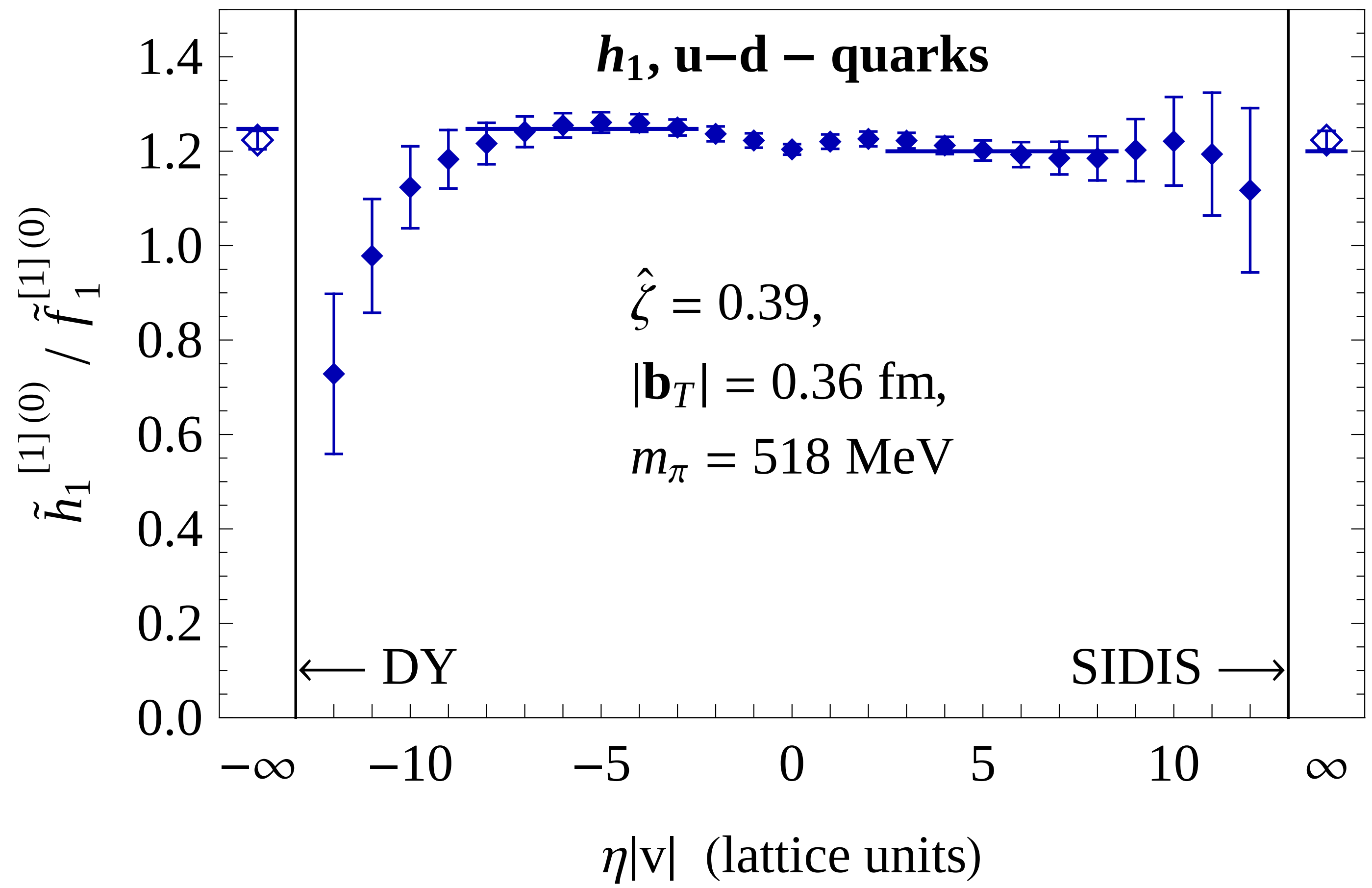
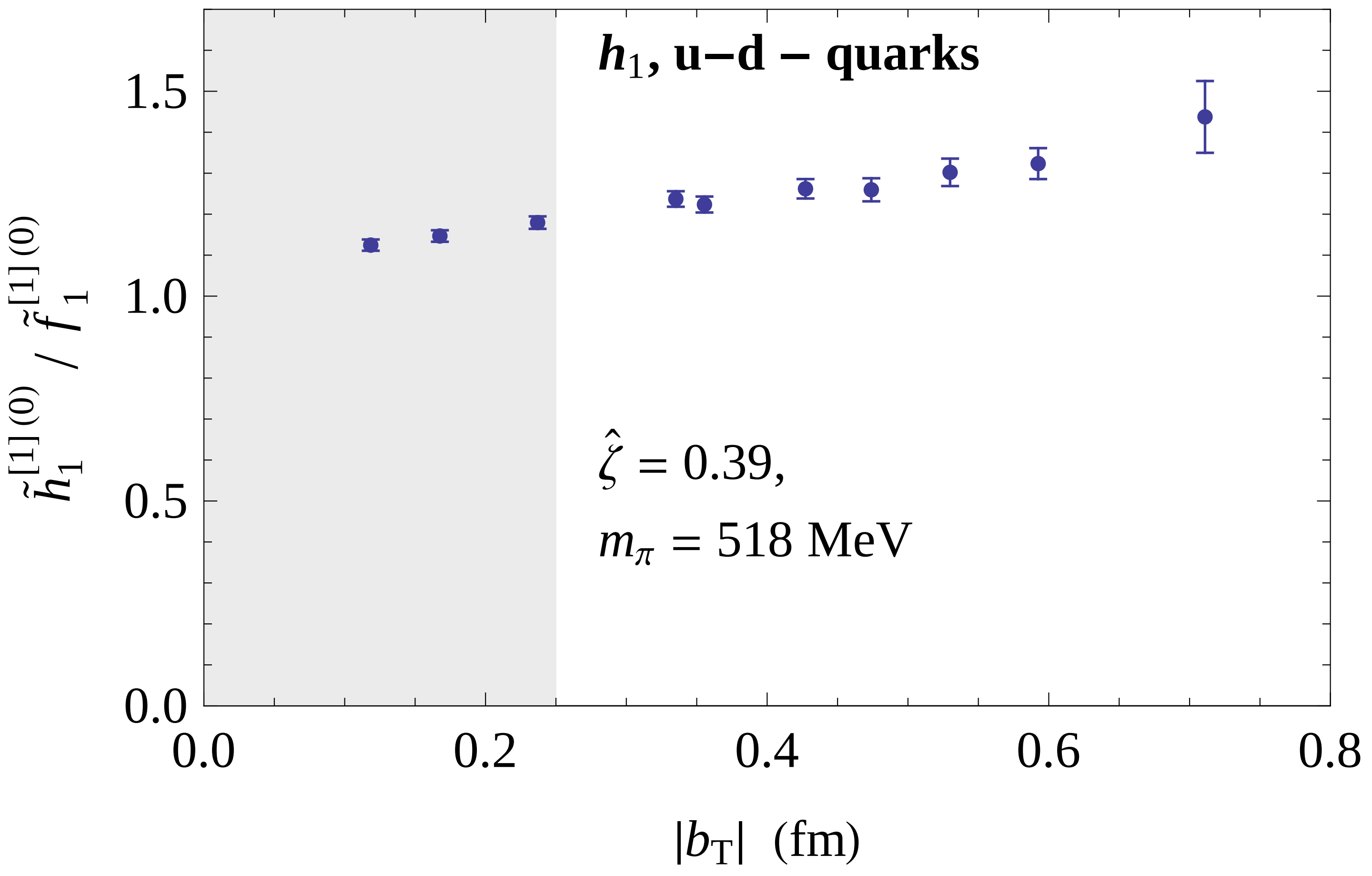


Results: Transversity

Dependence of SIDIS/DY limit on $|b_T|$



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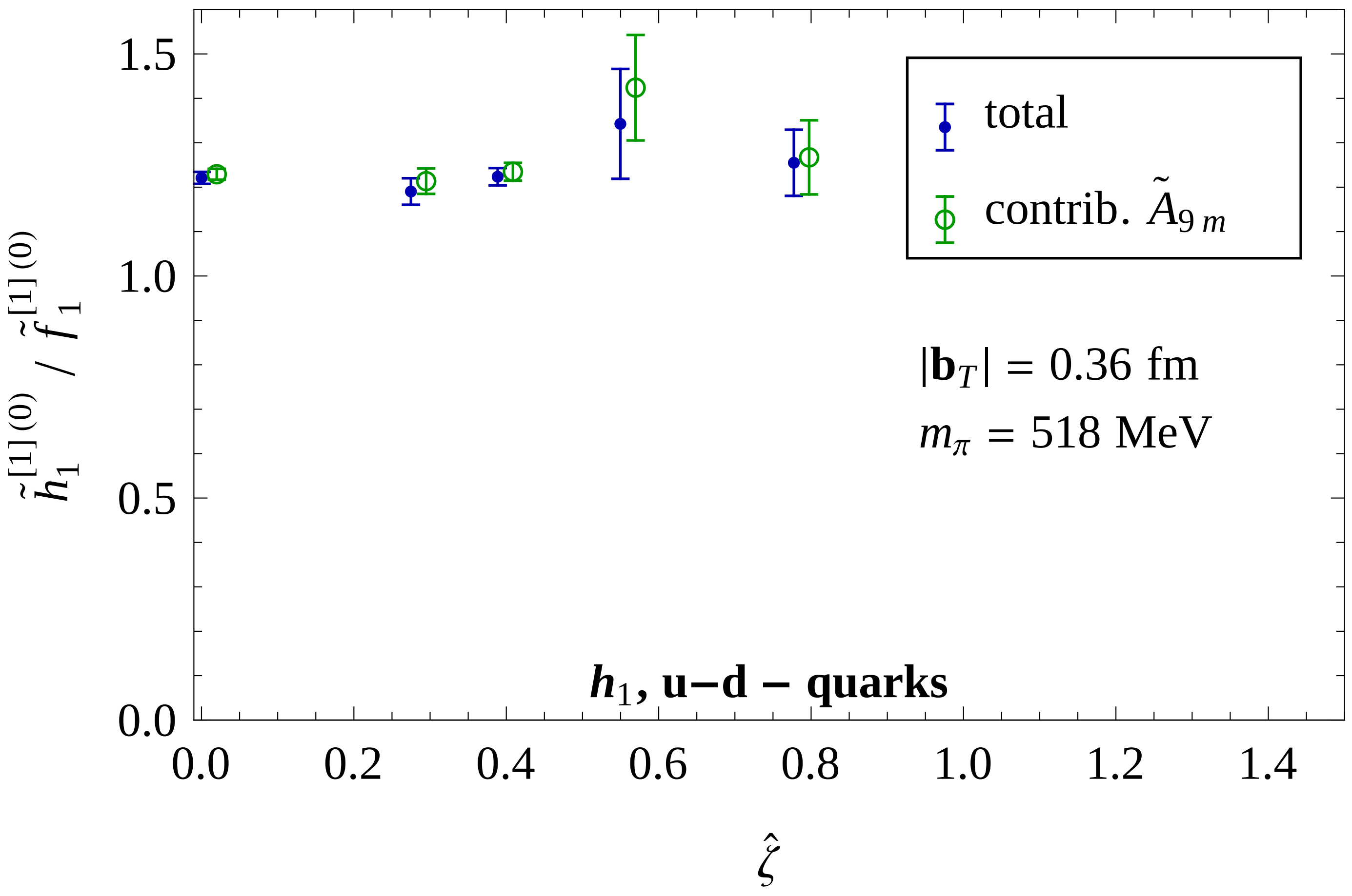
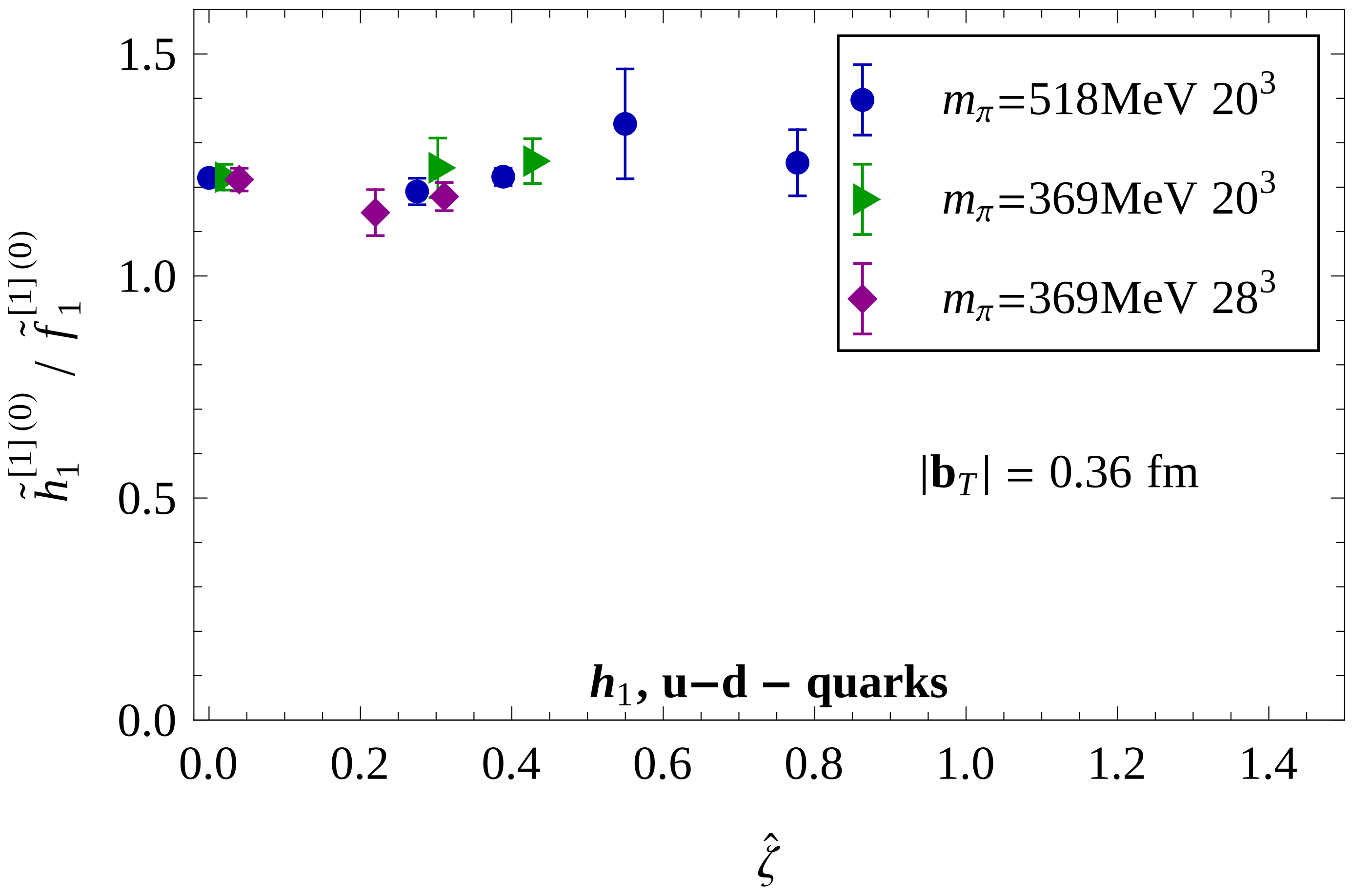
Dependence of SIDIS/DY limit on $\hat{\zeta}$



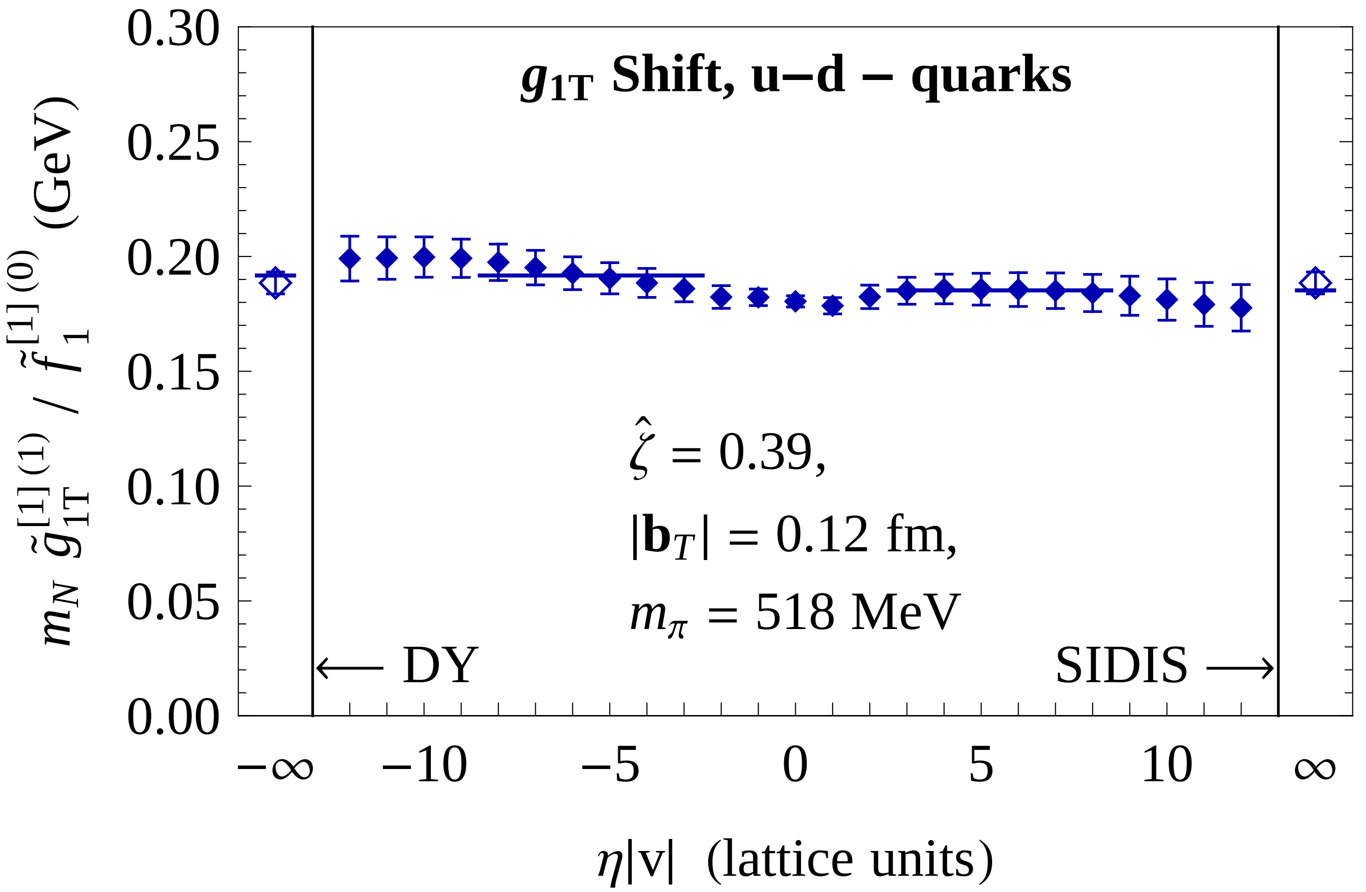
Results: Transversity

Dependence of SIDIS/DY limit on $\hat{\zeta}$, all three ensembles



Results: g_{1T} worm gear shift

Dependence on staple extent; sequence of panels at different $|b_T|$



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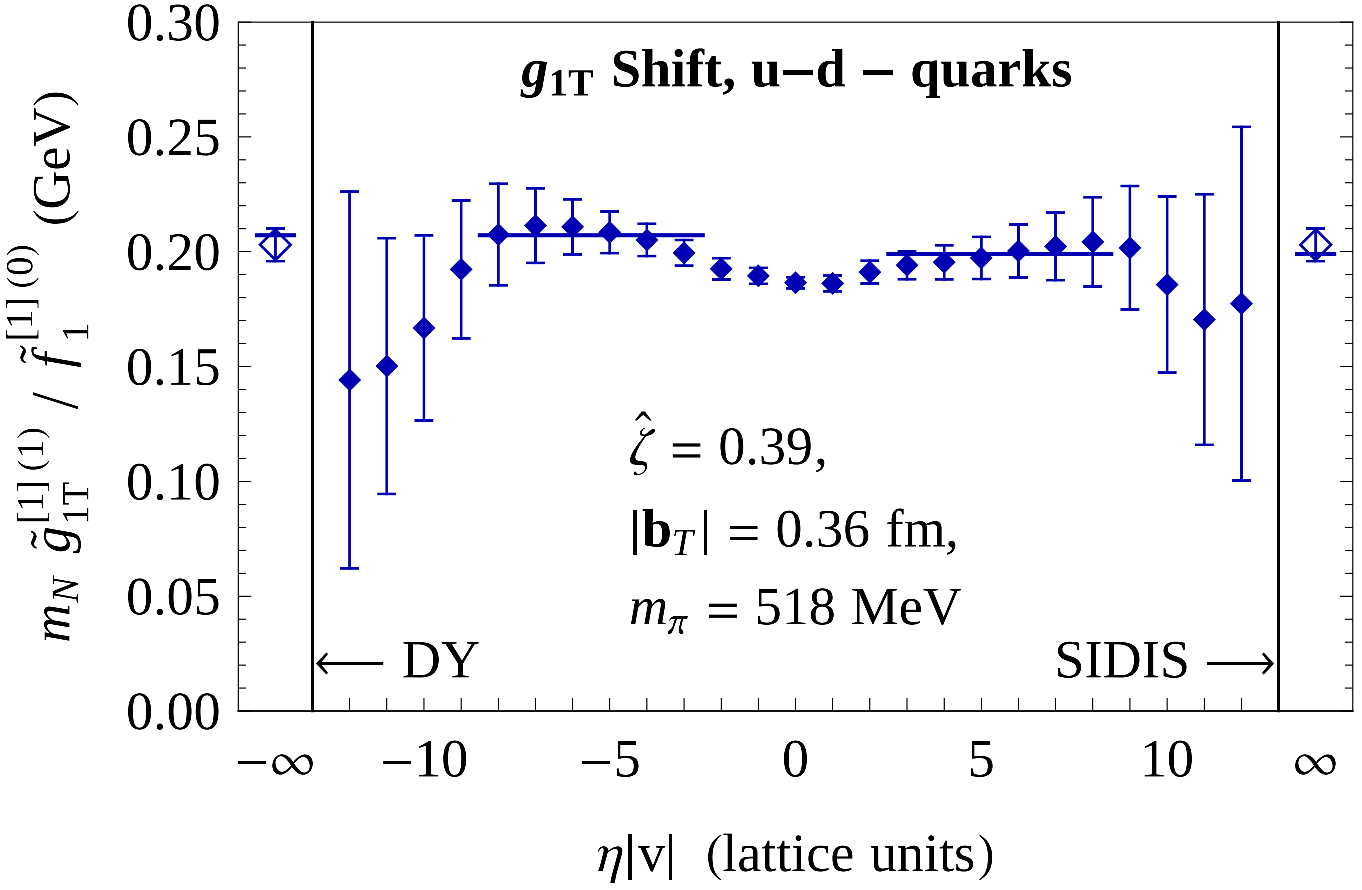

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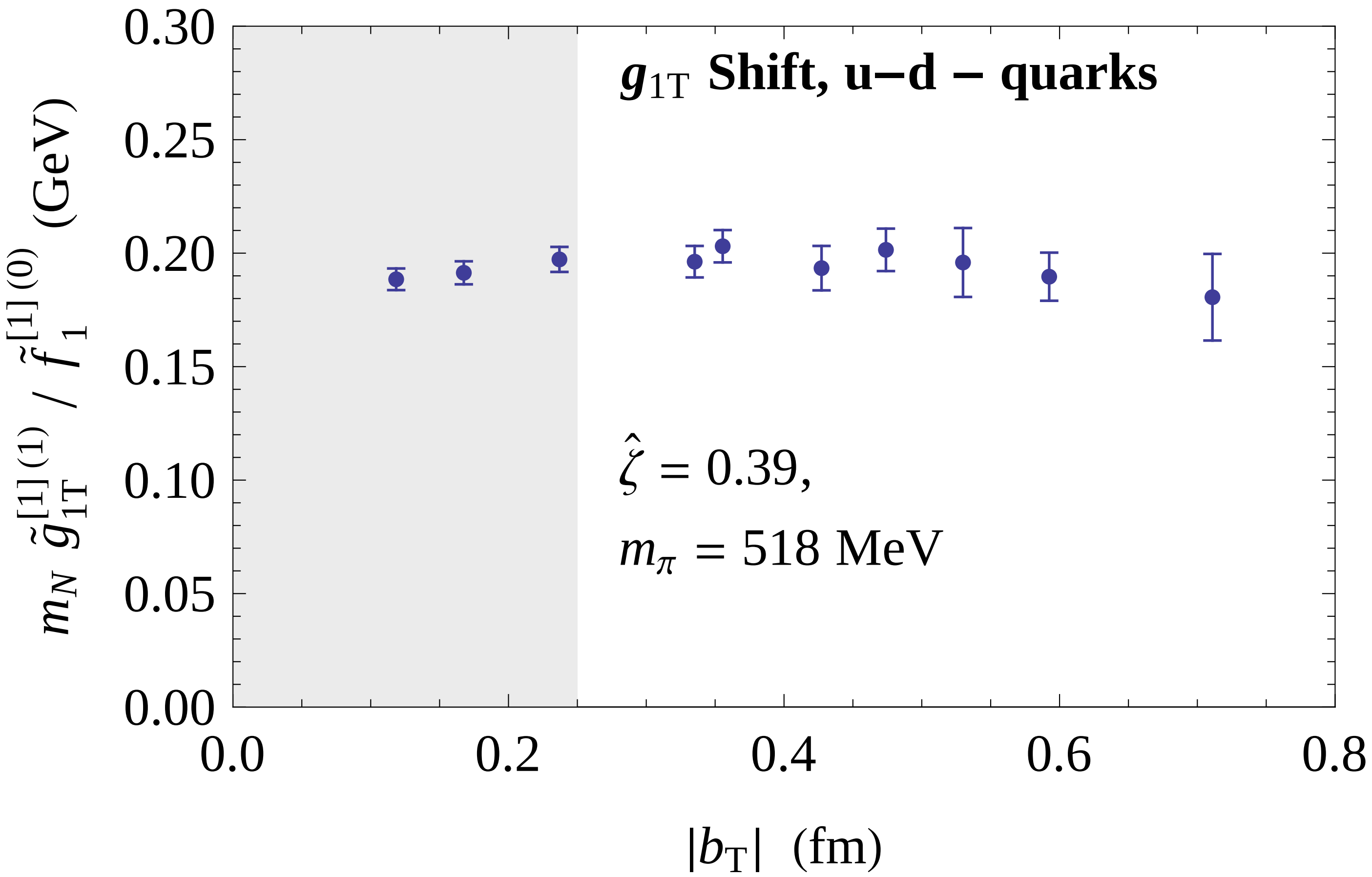

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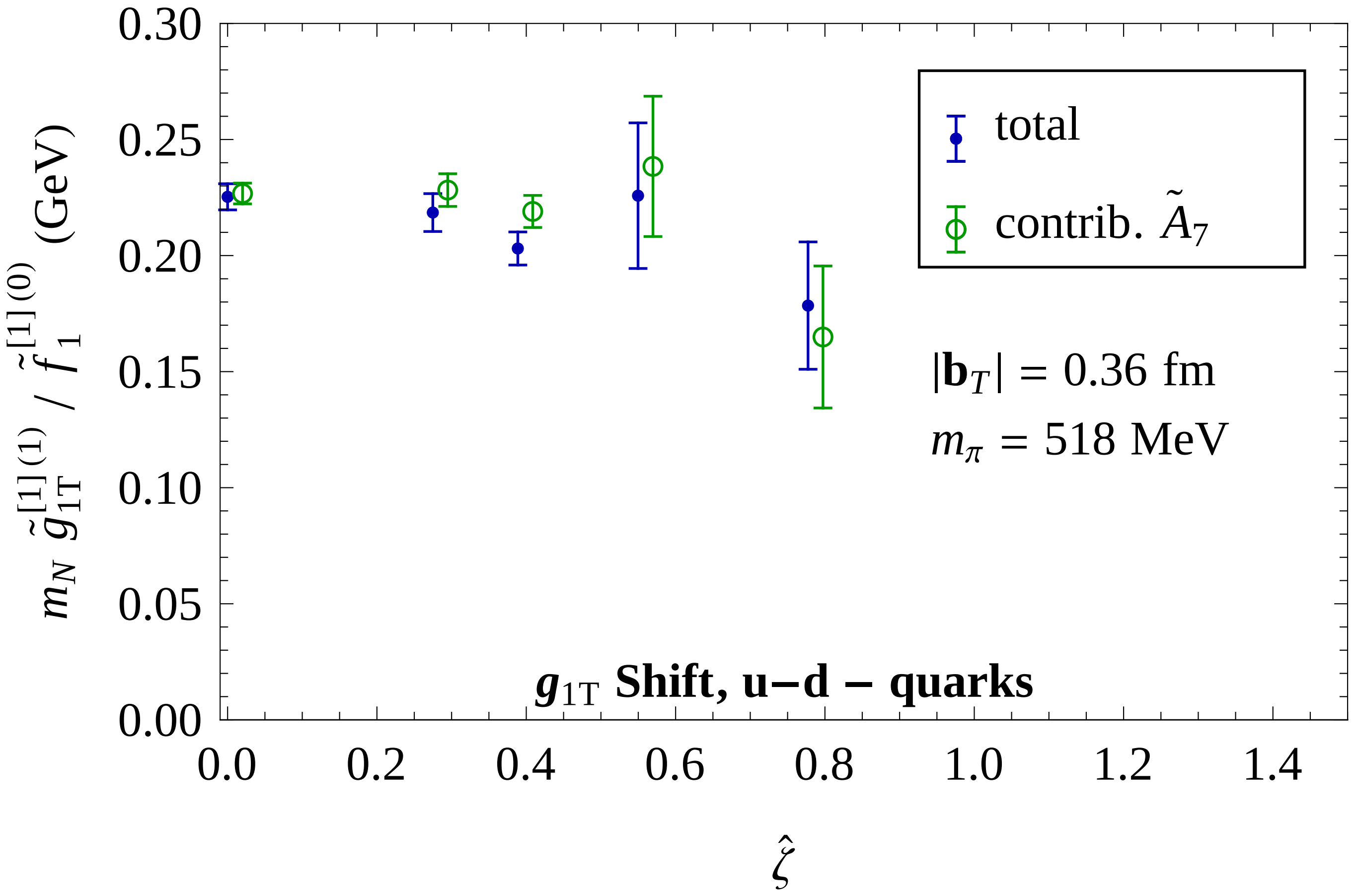

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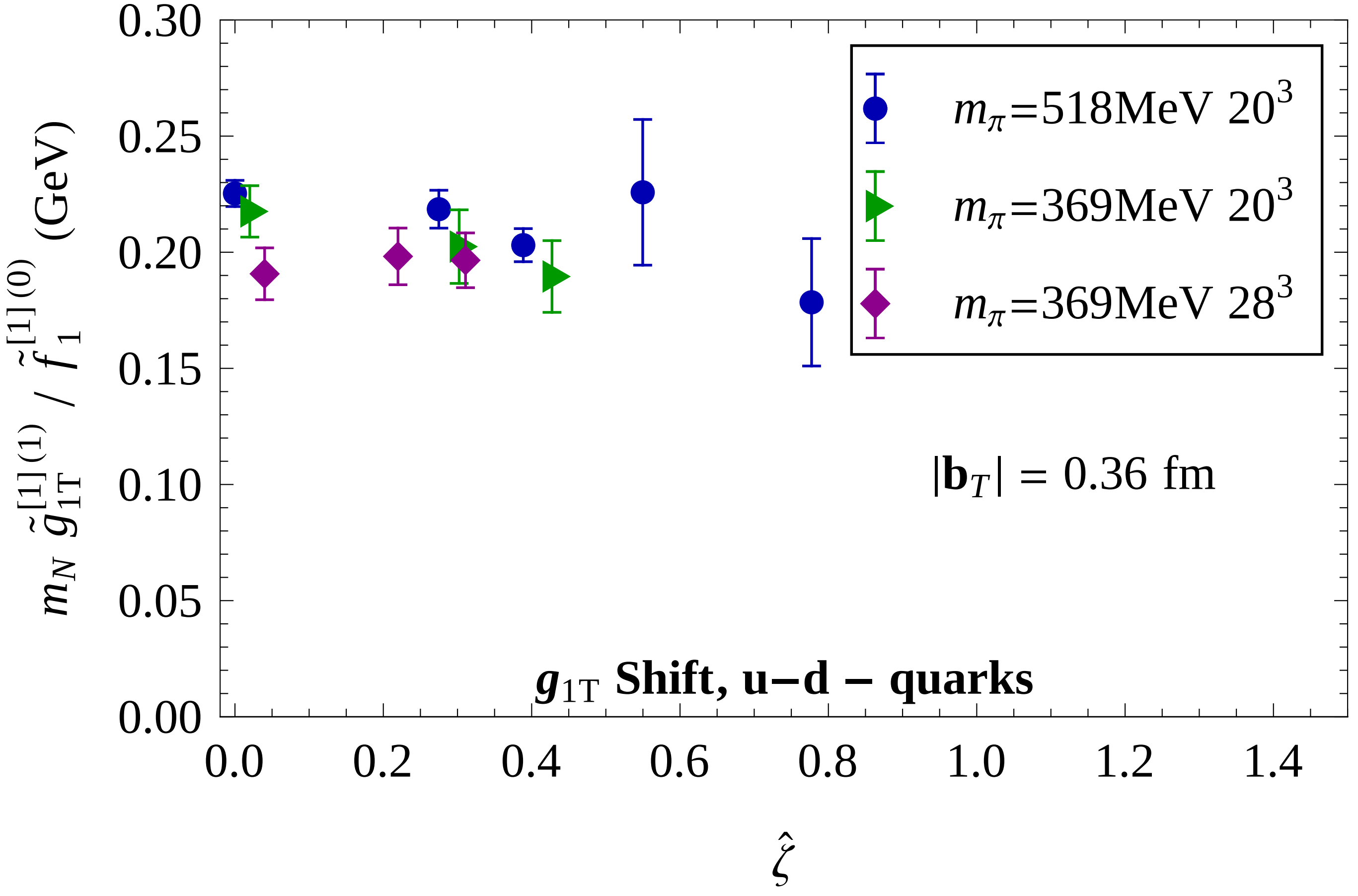

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