Anomalous processes and leading logarithms

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Chiral dynamics, JLab, 12/8/6

Outline:

- Anomalous processes
- Leading logarithms

Chiral perturbation theory

- low energy effective field theory of QCD
- degrees of freedom: goldstone bosons of spontaneous symmetry breakdown of chiral symmetry
- these are: pions, kaons and eta
- dimensional counting
- naively we would end up only with an even sector
- even sector known up to NNLO $\Leftrightarrow$ lagrangian $\mathcal{L}$ up to p^6
- $\phi \leftrightarrow -\phi$ in this $\mathcal{L}$: however this is not the symmetry of original QCD
- we know it exists $K^+K^- \to 3\pi$
- $\Rightarrow$ odd intrinsic parity sector

Chiral perturbation theory: odd sector

- existence of the chiral anomaly for the axial current
- The form of anomaly is determined by 'gauging' external fields (Wess-Zumino construction ['71]), explicitly for $SU(2)$:
  v and a are gauge fields of

$$L = R = 1 + i\alpha, \quad \text{and} \quad L^\dagger = R = 1 + i\beta$$

$$\delta S\{v,a,s,p,\theta\} = -\frac{N_C}{16\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\langle\hat{\beta}(\hat{v}_{\mu\nu} + i[\hat{a}_\mu,\hat{a}_\nu])\rangle\langle v_{\rho\sigma}\rangle$$

$$\mathcal{L}_{\mathrm{WZW}} = -\frac{N_C}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\left\{\langle U^\dagger\hat{r}_\mu U\hat{l}_\nu - \hat{r}_\mu\hat{l}_\nu + i\Sigma_\mu(U^\dagger\hat{r}_\nu U + \hat{l}_\nu)\rangle\langle v_{\rho\sigma}\rangle + \tfrac{2}{3}\langle\Sigma_\mu\Sigma_\nu\Sigma_\rho\rangle\langle v_\sigma\rangle\right\}$$

with $\Sigma_\mu = U^\dagger\partial_\mu U, \quad \hat{r}_\mu = \hat{v}_\mu + \hat{a}_\mu, \quad \hat{l}_\mu = \hat{v}_\mu - \hat{a}_\mu.$

- similarly for $SU(3)$
- however, there we have also pure odd-n-'pion'-vertex (as $2K3\pi$)

Calculation within 2-flavour ChPT

NLO odd-intrinsic Lagrangian is given by [Bijnens, Girlanda, Talavera '02] (for a different basis see [Ebertshauser, Fearing, Scherer '02])

$$\mathcal{L}_6^W = \sum_{i=1}^{13} c_i^W o_i^W\,, \qquad c_i^W = c_i^{Wr} + \eta_i (c\mu)^{d-4}\Lambda\,,$$

monomial (o_i^W)	i 2-flavour	$384\pi^2 F^2 \eta_i$
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{-\mu\nu}, u_\alpha u_\beta]\rangle$	1	0
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_- \{f_{+\mu\nu}, u_\alpha u_\beta\}\rangle$	2	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{+\mu\nu} f_{+\alpha\beta}\rangle$	3	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{-\mu\nu} f_{-\alpha\beta}\rangle$	4	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{+\mu\nu}, f_{-\alpha\beta}]\rangle$	5	0
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle\chi_- u_\alpha u_\beta\rangle$	6	$-5N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\chi_-\rangle$	7	$4N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\rangle\langle\chi_-\rangle$	8	$-2N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle h_{\gamma\nu} u_\alpha u_\beta\rangle$	9	$2N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle f_{-\gamma\nu} u_\alpha u_\beta\rangle$	10	$-6N_C$
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} h_{\gamma\beta}\rangle$	11	$4N_C$
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} f_{-\gamma\beta}\rangle$	12	0
$\epsilon^{\mu\nu\alpha\beta}\langle \nabla_\gamma f_{+\gamma\mu}\rangle\langle f_{+\nu\alpha} u_\beta\rangle$	13	$-4N_C$

n.b. it depends on the form of $\mathcal{L}_4$ [KK, Novotny 02], [Ananth.,Moussallam 02]

Small selection of applications

recent (< 10y) theoretical works:

- overview of phenomenology [Strandberg,Bijnens '02]
- two-photon decays $\pi^0/(\eta)$ [Goity,Bernstein,Holstein'02], [Borasoy,Nissler '04], [KK,Moussallam '09], [KK,Bijnens '10]
- $K_{\ell 2}\gamma$, $K_{\ell 3}\gamma$, $K_{\ell 4}$ [Gasser,Kubis,Paver,Verbeni'05], [Gerard,Smith,Trine'05], [Cirigliano et al. '12]
- $\pi^0 \to e^+e^-\gamma$ [KK,Knecht,Novotny]
- $\pi^+ \to e^+\nu\gamma$ [Mateu, Portoles '07],[Unterdorfer,Pichl'08]
- τ decays [Gómez Dumm et al.'10], [Guo,Roig '10], [Gonzalez-Alonso et al.'08]
- $\pi\gamma \to \pi\pi$ [Bijnens, KK, Lanz '12],[Giller et al.'05]
- $\eta \to \pi\pi\gamma$ [Borasoy,Nissler'04],[Stollenwerk et al.'12]
- $\eta \to 4\pi$ [Guo,Kubis,Wirzba'12]

\+ resonances: e.g. [Ruiz-Femenia,Pich,Portoles'03], [KK,Novotny'11]

\+ important application: $g-2$

\+ recently, important for holographic theories: e.g.[Colangelo,Sanz-Cillero,Zuo '12]

$\pi^0 \to \gamma\gamma$ at NNLO, result ([KK,Moussallam'09])

$$A_{NNLO} = \frac{e^2}{F_\pi}\Big\{\frac{1}{4\pi^2}$$
$$+\frac{16}{3}m_\pi^2(-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr}) + \frac{64}{9}B(m_d-m_u)(5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr})$$
$$-\frac{M^4}{24\pi^2F^4}\left(\frac{1}{16\pi^2}L_\pi\right)^2 + \frac{M^4}{16\pi^2F^4}L_\pi\Big[\frac{3}{256\pi^4} + \frac{32F^2}{3}\left(2c_2^{Wr} + 4c_3^{Wr} + 2c_6^{Wr} + 4c_7^{Wr} - c_{11}^{Wr}\right)\Big]$$
$$+\frac{32M^2B(m_d-m_u)}{48\pi^2F^4}L_\pi\left[-6c_2^{Wr} - 11c_3^{Wr} + 6c_4^{Wr} - 12c_5^{Wr} - c_7^{Wr} - 2c_8^{Wr}\right]$$
$$+\frac{M^4}{F^4}\lambda_+ + \frac{M^2B(m_d-m_u)}{F^4}\lambda_- + \frac{B^2(m_d-m_u)^2}{F^4}\lambda_{--}\Big\}$$

$$\lambda_+ = \frac{1}{\pi^2}\left[-\frac{2}{3}d_+^{Wr}(\mu) - 8c_6^r - \frac{1}{4}(l_4^r)^2 + \frac{1}{512\pi^4}\left(-\frac{983}{288} - \frac{4}{3}\zeta(3) + 3\sqrt{3}\,\mathrm{Cl}_2(\pi/3)\right)\right]$$
$$+\frac{16}{3}F^2\left[8l_3^r(c_3^{Wr} + c_7^{Wr}) + l_4^r(-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr})\right]$$
$$\lambda_- = \frac{64}{9}\left[d_-^{Wr}(\mu) + F^2 l_4^r\left(5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr}\right)\right]$$
$$\lambda_{--} = d_{--}^{Wr}(\mu) - 128F^2 l_7(c_3^{Wr} + c_7^{Wr})\ .$$

- 4 LECs in 2 combinations of NLO
- additional 4 LECs in 3 combinations of NNLO

Is it at all possible to make some reliable prediction?

$\pi^0 \to \gamma\gamma$: modified counting

- Use of $SU(3)$ phenomenology via $c_i^{Wr} \leftrightarrow C_i^{Wr}$ connection (based on [Gasser, Haefeli, Ivanov, Schmid '07,'08])

$$c_i^{Wr} = \frac{\alpha_i}{m_s} + \left(\beta_i + \gamma_{ij} C_j^{Wr} + \delta_i \ln \frac{B_0 m_s}{\mu^2}\right) + O(m_s)$$

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- implementation of modified counting

$$m_u, m_d \sim O(p^2) \quad \text{and} \quad m_s \sim O(p)$$

Result:

$$A_{NNLO}^{mod} = \frac{e^2}{F_\pi}\Bigg\{ \frac{1}{4\pi^2} - \frac{64}{3} m_\pi^2 C_7^{Wr} + \frac{1}{16\pi^2}\frac{m_d - m_u}{m_s}\Big[1 - \frac{3}{2}\frac{m_\pi^2}{16\pi^2 F_\pi^2} L_\pi\Big]$$
$$+ 32B(m_d - m_u)\Bigg[\frac{4}{3} C_7^{Wr} + 4 C_8^{Wr}\Big(1 - 3\frac{m_\pi^2}{16\pi^2 F_\pi^2} L_\pi\Big)$$
$$- \frac{1}{16\pi^2 F_\pi^2}\Big(3L_7^r + L_8^r - \frac{1}{512\pi^2}(L_K + \frac{2}{3}L_\eta)\Big)\Bigg] - \frac{1}{24\pi^2}\left(\frac{m_\pi^2}{16\pi^2 F_\pi^2} L_\pi\right)^2 \Bigg\}$$

$\pi \to \gamma\gamma$: Phenomenology

- $F_\pi = 92.22 \pm 0.07$ MeV (using updated value of V_{ud} [Towner, Hardy'08])
- $\frac{m_d - m_u}{m_s} = (2.29 \pm 0.23)\,10^{-2}$ (using quark mass ratio (from lattice), pseudo-scalar meson masses, R from $\eta \to 3\pi$)
- $B(m_d - m_u) = (0.32 \pm 0.03)\,M_{\pi^0}^2$
- $3L_7 + L_8^r(\mu) = (0.10 \pm 0.06)\,10^{-3} \quad (\mu = M_\eta)$ (from pseudo-scalar meson masses formula [Gasser, Leutwyler '85])
- $C_7^W = 0$ (more precisely $C_7^W \ll C_8^W$, motivated by simple resonance saturation)
- $C_8^W = (0.58 \pm 0.2)10^{-3}$GeV^{-2} (from $\eta \to 2\gamma$)

result $\Gamma_{\pi^0 \to 2\gamma} = (8.09 \pm 0.11)$eV

pion decay constant from $\pi \to \gamma\gamma$

- $\Gamma_{\pi^0\to 2\gamma}$: depends on V_{ud}: update of V_{ud} changes also F_π
- based on SM; deviation from standard $V - A$ leads to an effective $\hat{F}_\pi$ [Stern et aliæ'08]

$$F_\pi^2 = \hat{F}_\pi^2(1+\epsilon), \quad \text{with} \qquad \epsilon \sim V_R^{ud}/V_L^{ud}$$

- connection between F_π and F_{π^0} tiny [KK,Moussallam '09]

$$\left.\frac{F_{\pi^+}}{F_{\pi^0}}\right|_{QCD} - 1 = \frac{B^2(m_d-m_u)^2}{F_\pi^4}\left[-16\, c_9^r(\mu) - \frac{l_7}{16\pi^2}\left(1+\log\frac{m_\pi^2}{\mu^2}\right)\right]$$
$$\simeq 0.7\times 10^{-4}\ .$$

- $\Rightarrow$ one can thus use $\pi^0 \to \gamma\gamma$ for determination of F_π:

$$F_\pi \approx F_{\pi^0} = 93.85 \pm 1.3(\text{exp.}) \pm 0.6(\text{theory})\,\text{MeV} = 93.85 \pm 1.4\,\text{MeV}$$

- n.b. $\hat{F}_\pi = 92.22(7) \Rightarrow \epsilon \approx 3 - 4\%$ 1σ significance for right-handed currents

Too complicated?

Too complicated?

one loop only in the next...

Renormalizable theories

- we calculate e.g. $F(M)$:

$$F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + \dots$$
$$= \alpha + \alpha^2 f_1^1 L + \alpha^2 f_0^1 + \alpha^3 f_2^2 L^2 + \alpha^3 f_1^2 L + \alpha^3 f_0^2 + \dots$$

- where we have defined $L \equiv \log(\mu/M)$
- renormalization condition $\mu \frac{dF}{d\mu} = 0$
- non-trivial dependence on α

$$\mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \dots$$

- β_0 obtained from 1-loop diagrams
- renormalization condition $\Rightarrow$

$$f_1^1 = -\beta_0,\ f_2^2 = \beta_0^2,\ f_3^3 = -\beta_0^3 \quad \Rightarrow \quad F\big|_{LL} = \frac{\alpha}{1 + \alpha\beta_0 L}$$

Non-renormalizable theories

What are the Leading Logarithms (LL)?

- We calculate e.g. $F(M)$:

$$F = F_0 + \underline{F_1^1 L} + F_0^1 + \underline{F_2^2 L^2} + F_1^2 L + F_0^2 + \dots$$

- where we have defined $L \equiv \log(\mu/M)$

Why they are special?

- they are parameter-free
- to all orders from one-loop diagrams only (based on [Weinberg '79], [Büchler, Colangelo'03])

$O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model

$$\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi\,.$$

- explicit + spontaneous symmetry breaking

$$\langle \Phi^T \rangle = (1\;0\ldots0) \qquad \chi^T = (M^2\;0\ldots0)$$

- we have N Goldstone bosons: ϕ
- $N=3$ equivalent to two-flavour ChPT

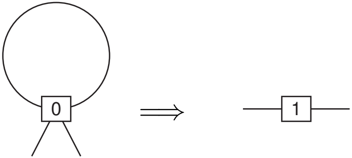
$O(N)$ sigma model: physical mass M_π

LL for the physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- $\mathcal{L}_{n\sigma} \Rightarrow \boxed{0}$
- mass: two-point function
- schematically at LO:

$\Leftrightarrow \quad M_\pi = M$

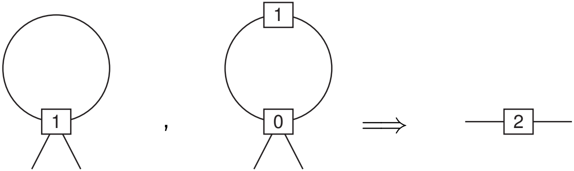
- NLO



$O(N)$ sigma model: physical mass M_π

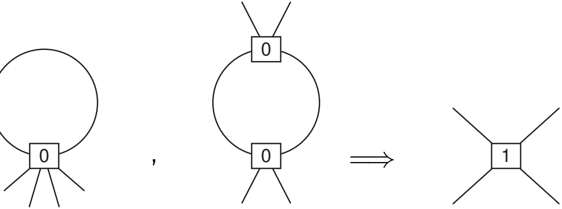
LL for physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- NNLO



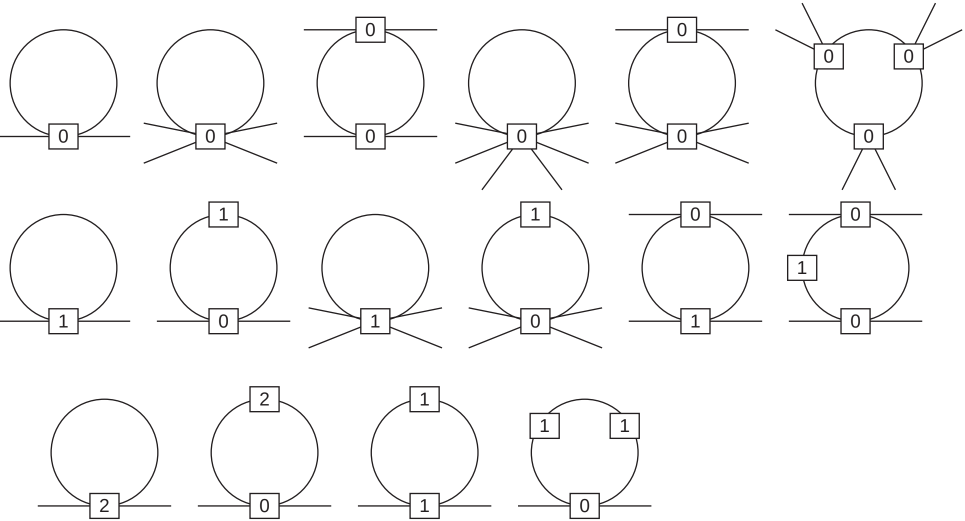
(1)

- we have to calculate first:

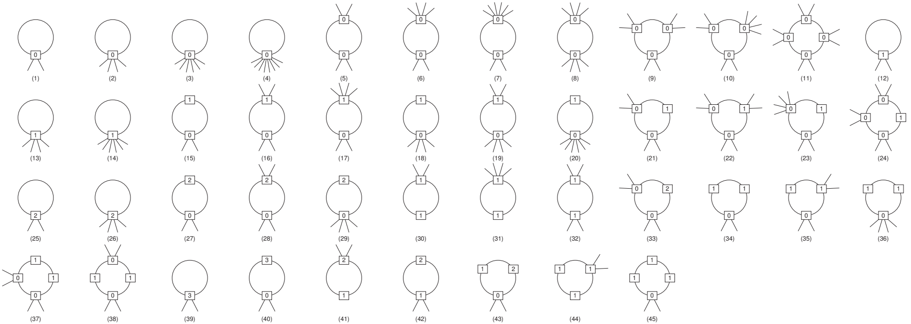


(2)

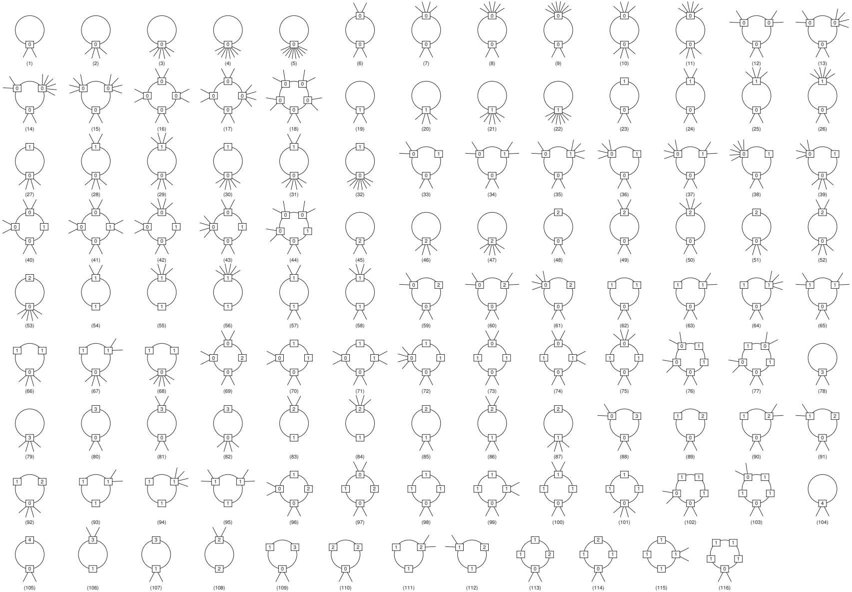
mass up-to 3-loop order



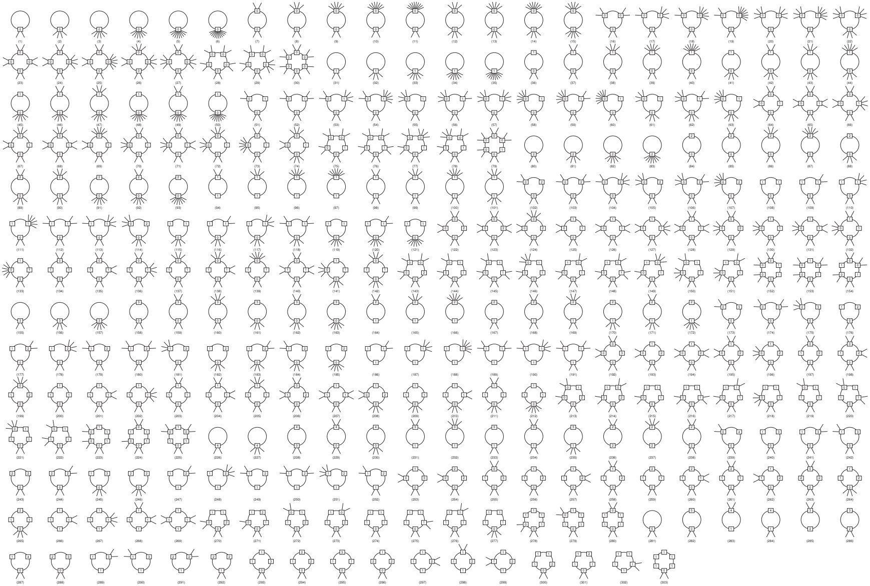
mass up-to 4-loop order



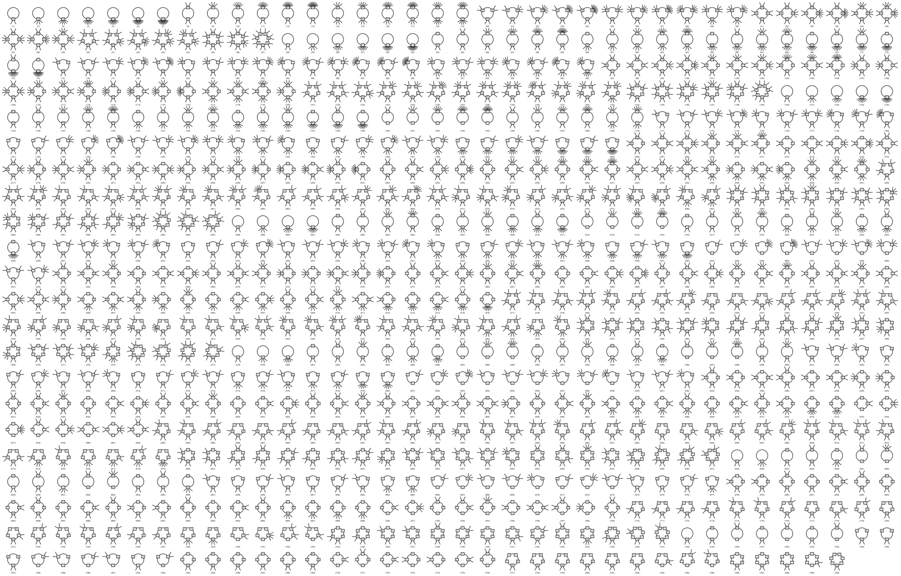
mass up-to 5-loop order



mass up-to 6-loop order



mass up-to 7-loop order



$O(N)$ sigma model: physical mass M_π, results

LL for physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- # of diagrams: $1, 5, 16, 45, 116, 303, 790, \ldots$
- $M_\pi^2 = M^2(1 + a_1 L_M + a_2 L_M^2 + \ldots)$
  $L_M = M^2/(16\pi^2 F^2)\log(\mu^2/M^2)$

i	a_i for $N=3$	a_i for general N
1	$-1/2$	$1 - 1/2\ N$
2	17/8	$7/4 - 7/4\ N + 5/8\ N^2$
3	$-103/24$	$37/12 - 113/24\ N + 15/4\ N^2 - N^3$
4	24367/1152	$839/144 - 1601/144\ N + 695/48\ N^2 - 135/16\ N^3 + 231/128\ N^4$
5	$-8821/144$	$33661/2400 - 1151407/43200\ N + 197587/4320\ N^2 - 12709/300\ N^3$ $+6271/320\ N^4 - 7/2\ N^5$
6	$\frac{1922964667}{6220800}$	$158393809/3888000 - 182792131/2592000\ N + 1046805817/7776000\ N^2$ $-17241967/103680\ N^3 + 70046633/576000\ N^4 - 23775/512\ N^5$ $+7293/1024\ N^6$
7*	$-\frac{1804453729667}{1714608000}$	$1098817478897/8573040000 - 286907006651/1428840000\ N$ $+4533157401977/11430720000\ N^2 - 1986536871797/3429216000\ N^3$ $+436238667943/762048000\ N^4 - 7266210703/21168000\ N^5$ $+99977/896\ N^6 - 15\ N^7$

* new

$O(N)$ sigma model: physical mass M_π, verification

Some cross-check? yes

- different parameterizations, (we have used 5) e.g.

$$\Phi_1 = \begin{pmatrix} \sqrt{1 - \frac{\phi^T\phi}{F^2}} \\ \frac{\phi}{F} \end{pmatrix}, \qquad \Phi_4 = \begin{pmatrix} \cos\sqrt{\frac{\phi^T\phi}{F^2}} \\ \sin\sqrt{\frac{\phi^T\phi}{F^2}}\frac{\phi}{\sqrt{\phi^T\phi}} \end{pmatrix}$$

- full two-loop result for $N = 3$
  - for $N = 3$ we have $SU(2) \times SU(2)/SU(2)$
  - well known: [Colangelo '95], [Bürgi '96], [Bijnens et al. '97]
- limits
  - massless limit
  - large N limit

$O(N)$ sigma model: physical mass M_π, verification

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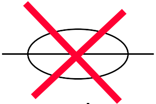
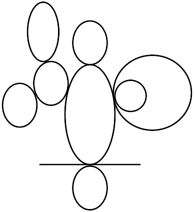
$O(N)$ sigma model: physical mass M_π, large N

[Bijnens, Carloni '09]

- powercounting: $F \sim \sqrt{N}$, $M^2 \sim 1$
- no lines shared between two loops



- as many closed flavour loops as there are loops
- only tadpoles $\Rightarrow$ cactus diagrams



$O(N)$ sigma model

already calculated [Bijnens, Carloni '09,'10], [Bijnens, KK, Lanz '12]

- mass M_π
- pion decay constant F_π
- $\phi\phi$ scattering, F_V, F_S

massless case [Kivel, Polyakov, Vladimirov '08,'09,'10,'10]

- $\phi\phi$ scattering, F_V, F_S
- recursive relations

N.B. knowing F_π enables the expansion both in $L \sim M^2/F^2 \times \log M$
and in $L_\pi \sim M_\pi^2/F_\pi^2 \times \log M_\pi$

$O(N)$ sigma model: vector formfactor

- for $\pi \to \gamma\gamma$ we will need even number of pions and one photon
- effectively hidden in F_V
- definition

$$\langle \phi^a(\mathbf{p}_f)|j^{cd}_{V,\mu} - j^{dc}_{V,\mu}|\phi^b(\mathbf{p}_i)\rangle = \left(\delta^{ac}\delta^{db} - \delta^{ad}\delta^{bc}\right) i(\mathbf{p}_f + \mathbf{p}_i)^\mu F_V\left[(\mathbf{p}_f - \mathbf{p}_i)^2\right] ,$$

- calculated already in [Bijnens, Carloni '10]
- problem in cross-check with massless case in [Polyakov et al]
- independent study using disp.relations ($\pi\pi$ partial amplitude) [Bijnens,KK,Lanz '12]

$$\mathrm{disc} F_V(s) = t^1_1 F_V(s)$$

- $\Rightarrow$ [Polyakov et al] mistake

$O(N)$ sigma model: vector formfactor

[Bijnens,Carloni '10],[Bijnens,KK,Lanz '12]

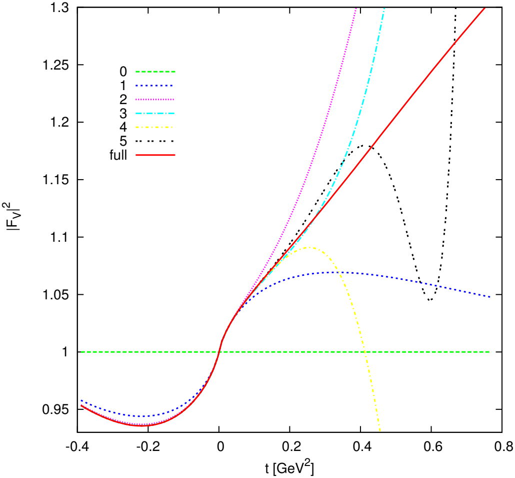
- calculated up to 5 loops
- chiral limit (see [Polyakov et al])
- closed form for NLL

$$F_V^{0NLL}(t) = 1 + \frac{1}{N} + \frac{4}{K_t N^2}\left[1 - \left(1 + \frac{2}{K_t N}\right)\log\left(1 + \frac{K_t N}{2}\right)\right]$$

$$K_t \equiv \frac{t}{16\pi^2 F^2}\log(-\mu^2/t)$$

$O(N)$ sigma model: vector formfactor

The expansions of the leading logarithms order by order for the vector form-factor in the chiral limit and next-to-large N limit ($F = 0.090$ GeV, $\mu = 0.77$ GeV and $N = 3$



Odd sector

$O(N)$ done only for $N = 3$ $(SU(2))$

1. $\pi\gamma \to \pi\pi$
2. $\pi \to \gamma\gamma$

Odd sector: 1. $\pi\gamma \to \pi\pi$

- $\pi^0\gamma \to \pi^0\pi^0$: C forbidden
- $\pi^-(p_1)\gamma(k) \to \pi^-(p_2)\pi^0(p_0)$ described by $VAAA$ box anomaly
- Mandelstam variables: s, t, u

$$s + t + u = 3M_\pi^2 + k^2$$

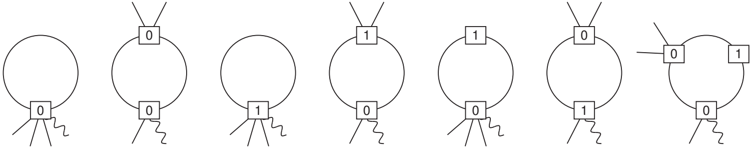
- due to symmetry, reasonable to take

$$\Delta_n = s^n + t^n + u^n$$

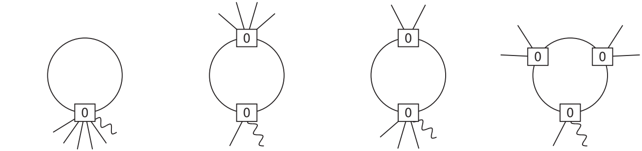
- $$F^{3\pi} = F_0^{3\pi} f(s,t,u), \qquad F_0^{3\pi} = \frac{e}{4\pi^2 F_\pi^3}$$

Odd sector: 1. $\pi\gamma \to \pi\pi$

The irreducible diagrams for the process $\pi\gamma \to \pi\pi$ up to two-loop level



auxiliary diagrams



Odd sector: 1. $\pi\gamma \to \pi\pi$

$$
\begin{aligned}
f^{LL}(s,t,u) = {} & 1 + L_{\mathcal{M}}\frac{1}{6}\left(3+\tilde{k}^2\right) + L_{\mathcal{M}}^2\frac{1}{72}(\tilde{k}^2-3)(\tilde{k}^2+33) \\
& + L_{\mathcal{M}}^3\frac{1}{1296}\left(90\tilde{\Delta}_3 - 640\tilde{\Delta}_2 - 8157 + 2105\tilde{k}^2 + 81\tilde{k}^4 + \tilde{k}^6\right) + L_{\mathcal{M}}^4\frac{1}{155520}\Big[-1532\tilde{\Delta}_4 \\
& + \tilde{\Delta}_3(88538 + 1890\tilde{k}^2) - \tilde{\Delta}_2(577760 + 12240\tilde{k}^2 + 540\tilde{k}^4) - 2433375 + 1296190\tilde{k}^2 \\
& + 57430\tilde{k}^4 + 480\tilde{k}^6 + 185\tilde{k}^8\Big] + L_{\mathcal{M}}^5\frac{1}{326592000}\Big[\tilde{\Delta}_5(13252156) \\
& - \tilde{\Delta}_4(160744570 + 518350\tilde{k}^2) + \tilde{\Delta}_3(1465187530 + 39593272\tilde{k}^2 + 247260\tilde{k}^4) \\
& - \tilde{\Delta}_2(6756522937 + 257781206\tilde{k}^2 + 11188776\tilde{k}^4 - 9160\tilde{k}^6) - 6498695163 \\
& + 12675091794\tilde{k}^2 + 801259373\tilde{k}^4 + 4780240\tilde{k}^6 + 2948600\tilde{k}^8 - 1832\tilde{k}^{10}\Big].
\end{aligned}
$$

remark: massless limit simple, no function found so-far

Phenomenology, 1. $\pi\gamma \to \pi\pi$

- little experimental information
- officially only experiment at Serpukhov

$$\pi^- + (Z, A) \to \pi^- + \pi^0 + (Z, A) \; . \qquad (3)$$

- Their analysis only includes events with low invariant mass of the final state pion pair ($\sqrt{s} < \sqrt{10} M_\pi$)
- the measurement indicates

$$\bar{F}^{3\pi}_{\text{exp}} = 12.9 \pm 0.9 \pm 0.5 \text{ GeV}^{-3}$$

- theoretical value $F^{3\pi}_0 = 9.8$ GeV^{-3}
- new activity (?)
  - via $\pi^- e^- \to \pi^- e^- \pi^0$ (SPS CERN '85) [Giller,Scherer et al. '05]
  - CLAS in CEBAF at JLab

Phenomenology, 1. $\pi\gamma \to \pi\pi$

- higher order corrections

$$F^{3\pi}(s,t,u) = F_0^{3\pi}(1 + f_0^{\text{EM}} + f_1^{\text{loop}} + f_1^{\text{ct}} + \ldots)\,.$$

- 1-loop corrections: [Bijnens, Bramon, Cornet '89]
- EM corrections: [Ametller, Knecht, Talavera '01]
- different models for counter-terms, e.g. hidden local symmetry (HLS), quark constituent model (QCM) [Strandberg '03], Schwinger-Dyson equation (SDE) [Jiang,Wang '10],
- The experimental extraction of the anomalous $\gamma 3\pi$ factor $F_0^{3\pi}$ (in GeV^{-3}) using various models for higher order corrections

	LO	f_0^{EM}	f_1^{loop}(LL)	f_1^{loop}	f_1^{ct} (HLS)	f_1^{ct} (QCM)	f_1^{ct} (SDE)
$F_0^{3\pi}$	12.9	12.3	12.0	11.8	11.3	10.1	11.9

Phenomenology, 1. $\pi\gamma \to \pi\pi$, LL

- The contributions from the leading logarithms up to sixth order to the experimental extraction of the anomalous $\gamma 3\pi$ factor $F_0^{3\pi}$ (in GeV^{-3}).

	LO	Δ1-LL	Δ2-LL	Δ3-LL	Δ4-LL	Δ5-LL
$F_0^{3\pi}$	12.9	-0.3	0.04	0.02	0.006	0.001

- energy dependence via some expansion, e.g. in s around $\pi\pi$ threshold

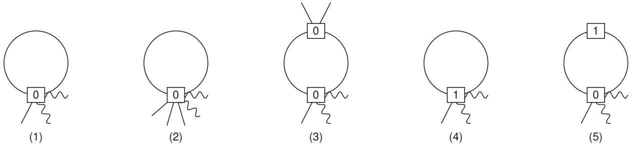
$$F^{3\pi}(s, t = 4M_\pi^2) = F_0^{3\pi}(1 + \alpha + \beta s/M_\pi^2 + O(s^2))$$

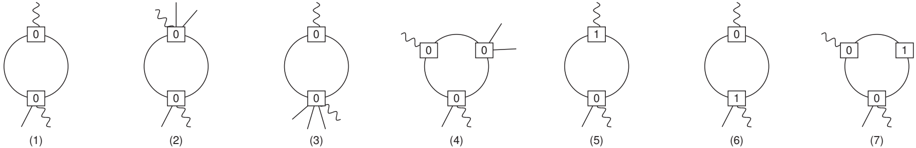
	α	β
c_1	$1/2$	0
c_2	$-11/8$	0
c_3	$-13367/1296$	$-775/648$
c_4	$-1414225/31104$	$-237877/25920$
c_5	$-14201792401/81648000$	$-4652736041/81648000$

Odd sector: 2. $\pi^0\gamma\gamma$

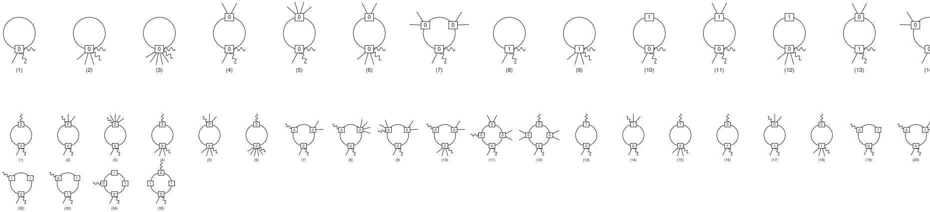
- the most important process of this sector
- important in normalization of π^0 decays
- topology complicated: two types of one-loop diagrams
- no logarithms at one-loop order [Donoghue et al'86], [Bijnens et al'88]
- true only for on-shell case
- logarithms at 2-loop order [KK,Moussallam'09]

Odd sector: 2. $\pi^0\gamma\gamma$, 2 loop

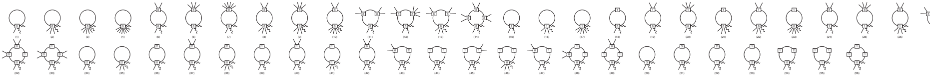


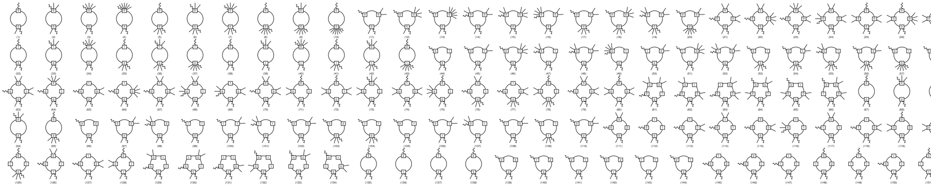


Odd sector: 2. $\pi^0\gamma\gamma$, 3 loop

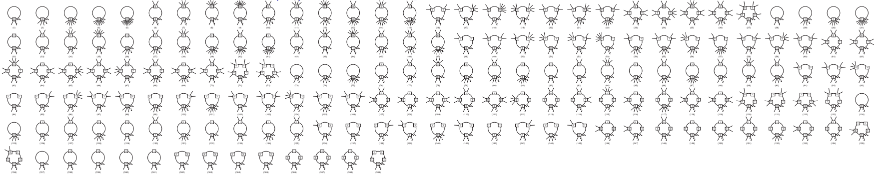


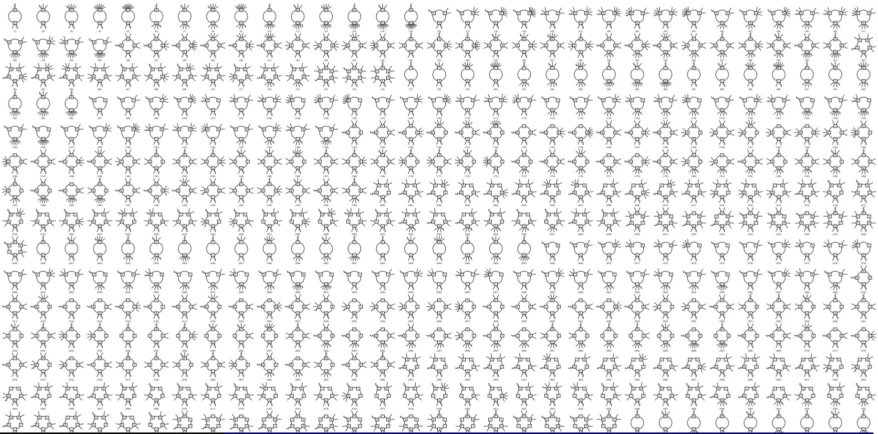
Odd sector: 2. $\pi^0\gamma\gamma$, 4 loop





Odd sector: 2. $\pi^0\gamma\gamma$, 5 loop





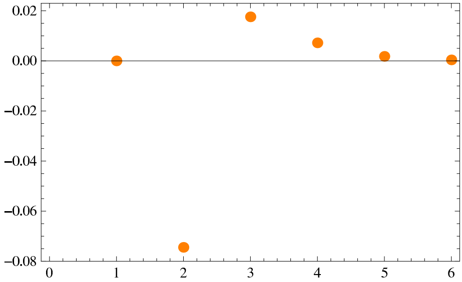
Odd sector: 2. $\pi^0\gamma\gamma$, result

$$F_{\pi\gamma\gamma}(k_1^2,k_2^2) = \frac{e^2}{4\pi^2 F_\pi} F_\gamma(k_1^2) F_\gamma(k_2^2) F_{\gamma\gamma}(k_1^2,k_2^2)\hat{F}\,.$$

$$\begin{aligned}
\hat{F} &= 1 - 1/6\, L_{\mathcal{M}}^2 + 5/6\, L_{\mathcal{M}}^3 + 56147/7776\, L_{\mathcal{M}}^4 + 446502199/11664000\, L_{\mathcal{M}}^5 \\
&\quad +65694012997/367416000\, L_{\mathcal{M}}^6\,, \\
F_\gamma(k^2) &= 1 + L_{\mathcal{M}}(1/6\,\tilde{k}^2) + L_{\mathcal{M}}^2(5/24\,\tilde{k}^2 + 1/72\,\tilde{k}^4) + L_{\mathcal{M}}^3(71/432\,\tilde{k}^2 + 1/24\,\tilde{k}^4 \\
&\quad +1/1296\,\tilde{k}^6) + L_{\mathcal{M}}^4(-24353/31104\,\tilde{k}^2 + 4873/10368\,\tilde{k}^4 - 2357/31104\,\tilde{k}^6 \\
&\quad +145/31104\,\tilde{k}^8) + L_{\mathcal{M}}^5(-548440741/81648000\,\tilde{k}^2 + 9793363/3024000\,\tilde{k}^4 \\
&\quad -32952389/54432000\,\tilde{k}^6 + 487493/13608000\,\tilde{k}^8 - 2069/10886400\,\tilde{k}^{10})\,, \\
F_{\gamma\gamma}(k_1^2,k_2^2) &= 1 + L_{\mathcal{M}}^3\tilde{k}_1^2\tilde{k}_2^2\frac{1}{72} + L_{\mathcal{M}}^4\tilde{k}_1^2\tilde{k}_2^2\big[-203/7776 + 29/10368(\tilde{k}_1^2+\tilde{k}_2^2) \\
&\quad +1/216(\tilde{k}_1^4+\tilde{k}_2^4) - 1/144\tilde{k}_1^2\tilde{k}_2^2\big] + L_{\mathcal{M}}^5\tilde{k}_1^2\tilde{k}_2^2\big[-5983633/10206000 \\
&\quad +46103/1632960(\tilde{k}_1^2+\tilde{k}_2^2) + 372113/11664000(\tilde{k}_1^4+\tilde{k}_2^4) \\
&\quad -211/5443200(\tilde{k}_1^6+\tilde{k}_2^6) - 394157/9072000\tilde{k}_1^2\tilde{k}_2^2 \\
&\quad -4/25515\tilde{k}_1^2\tilde{k}_2^2(\tilde{k}_1^2+\tilde{k}_2^2)\big]\,.
\end{aligned}$$

Phenomenology, 2. $\pi^0 \to \gamma\gamma$

Leading logarithm contribution of individual orders in percent of the leading order:

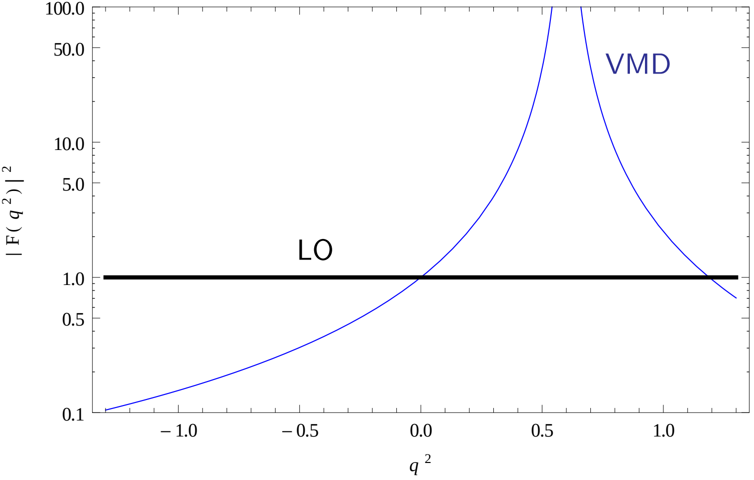


Adler-Lee-Treiman-Zee-Terentev theorem on triangle and box anomaly

$$F^{3\pi}(0,0,0) = \frac{1}{eF_\pi^2} F_{\pi\gamma\gamma}(0,0)$$

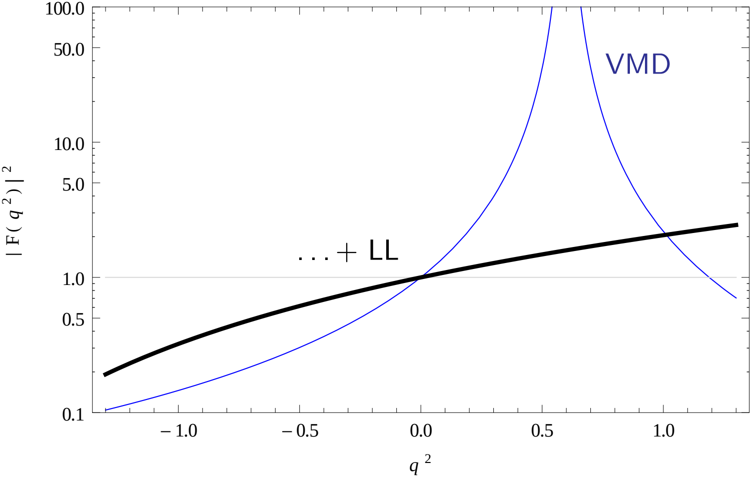
is valid up to 2-loop order for LL beyond the soft-photon limit

Phenomenology, 2. $\pi^0 \to \gamma^* \gamma$



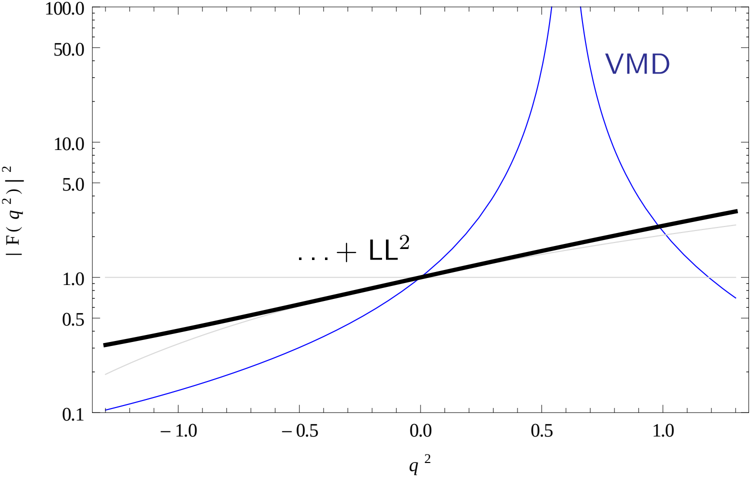
more info on formfactors: [MesonNet 1207.6556]

Phenomenology, 2. $\pi^0 \to \gamma^*\gamma$



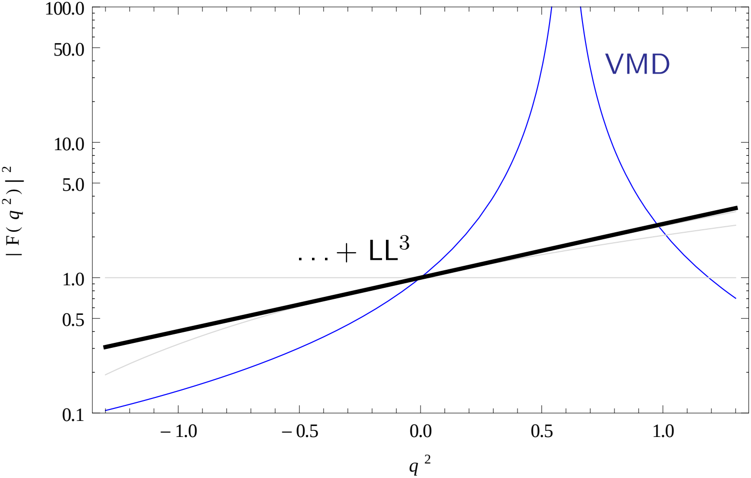
more info on formfactors: [MesonNet 1207.6556]

Phenomenology, 2. $\pi^0 \to \gamma^* \gamma$



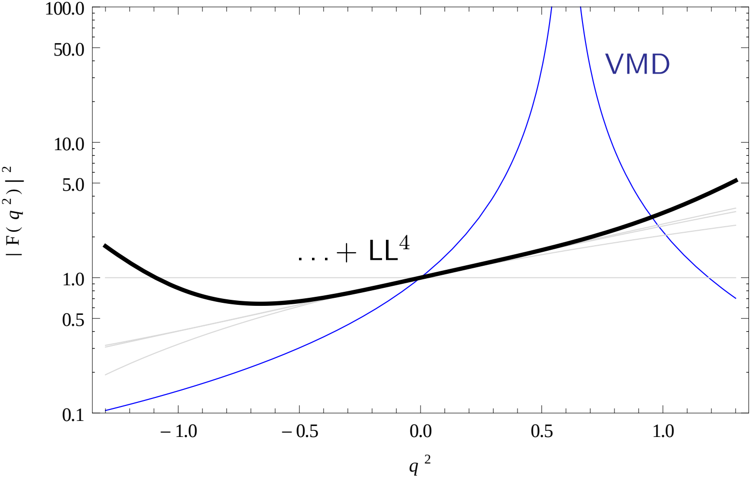
more info on formfactors: [MesonNet 1207.6556]

Phenomenology, 2. $\pi^0 \to \gamma^*\gamma$



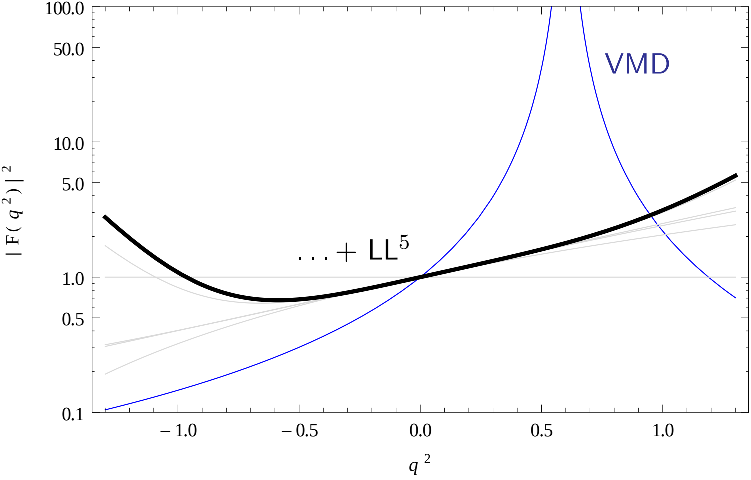
more info on formfactors: [MesonNet 1207.6556]

Phenomenology, 2. $\pi^0 \to \gamma^*\gamma$



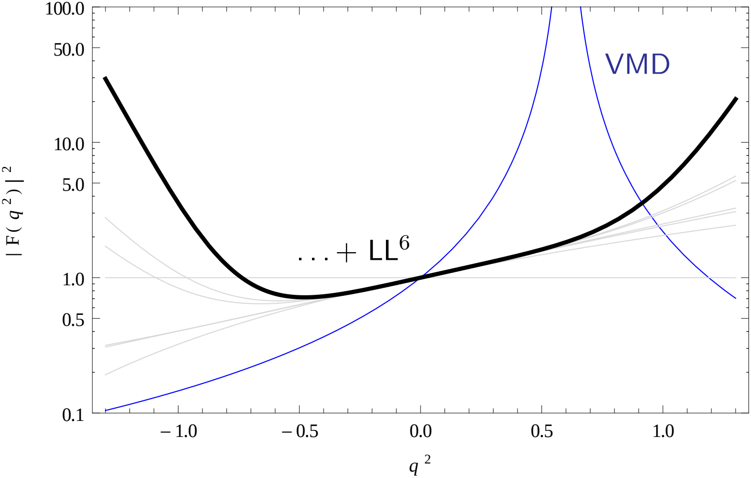
more info on formfactors: [MesonNet 1207.6556]

Phenomenology, 2. $\pi^0 \to \gamma^*\gamma$



more info on formfactors: [MesonNet 1207.6556]

Phenomenology, 2. $\pi^0 \to \gamma^*\gamma$



more info on formfactors: [MesonNet 1207.6556]

New project: $SU(N) \times SU(N)$

In collaboration with J.Bijnens and S.Lanz

- even sector almost finished
    - one example on next page
- odd sector slowly starting
- two physical cases: $N = 2$ and $N = 3$
- many checks available
- large N limit: planar diagrams now
- any resummation, any relation would be valuable

$SU(N)$ preliminary result: physical mass M_π

- \# of diagrams: $1, 5, 16, 45, 116, 303, 790, \ldots$
- $M_\pi^2 = M^2(1 + a_1 L_M + a_2 L_M^2 + \ldots)$
  $L_M = M^2/(16\pi^2 F^2)\log(\mu^2/M^2)$

i	a_i for $N=2$	a_i for $N=3$	a_i for general N
1	$-1/2$	$-1/3$	$-N^{-1}$
2	17/8	$27/8$	$-1/2 + 9/2N^{-2} + 3/8N^2$
3	$-103/24$	$-3799/648$	$-89/3N^{-3} + 19/3N^{-1} - 37/24N - 1/12N^3$
4	24367/1152	$146657/2592$	$+193/18 + 2015/8N^{-4} - 773/12N^{-2} + 121/288N^2$ $+41/72N^4$
5	$-8821/144$	$-27470059/186624$	$-38684/15N^{-5} + 6633/10N^{-3} - 59303/1080N^{-1}$ $-5077/1440N - 11327/4320N^3 - 8743/34560N^5$
6*	$\frac{1922964667}{6220800}$	$\frac{12902773163}{9331200}$	$205365409/972000 + 7329919/240N^{-6}$ $-1652293/240N^{-4} - 4910303/15552N^{-2}$ $-69368761/7776000N^2 + 14222209/2592000N^4$ $+3778133/3110400N^6$

* very preliminary