Determination of ChPT low energy constants from a precise description of $\pi\pi$ scattering threshold parameters

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Chiral Dynamics 2012, JLab

August 2012

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Purpose

Use a very recent **dispersive analysis of data***, in order to determine the **values of the $O(p^4)$ and $O(p^6)$ LECs** (low energy constants) appearing in the ChPT $\pi\pi$ scattering amplitudes.

We do it by fitting coefficients of the momentum expansion around threshold.

* R. Garcia-Martin, R. Kaminski, J. R. Pelaez, J. Ruiz de Elvira, F. J. Yndurain, Phys. Rev. **D83**, 074004 (2011).

Introduction

Chiral Perturbation Theory Weinberg, Gasser & Leutwyler

Low energy ($\ll 4\pi f_\pi \sim$1.2 GeV) effective theory of QCD with:

- DOF: $\pi \to$ pseudo-Goldstone bosons (NGB) of the spontaneous chiral symmetry breaking
- most general expansion in masses and momenta

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots$$

- parameters: Low Energy Constants (LECs)
  - absorbe loop divergencies
  - contain details of underlying dynamics of QCD
  - must be determined phenomenologically or from lattice calculations

Low Energy Constants (LECs) in $\pi\pi$ scattering

- Leading order $(\mathcal{O}(p^2))$
  - m_π, f_π
- Next-to-leading order $(\mathcal{O}(p^4))$
  - l_1, l_2, l_3, l_4
- Next-to-next-to-leading order $(\mathcal{O}(p^6))$
  - $b_1, b_2, b_3, b_4, b_5, b_6$

Threshold parameters

$\pi\pi$ scattering amplitudes decomposed in partial waves

$$F_{(I)}(s,t) = \frac{T_{(I)}(s,t)}{4\pi^2} = \frac{8}{\pi} \sum_\ell (2\ell+1)\, t_{\ell I}(s)\, P_\ell(\cos\theta)$$

$t_{\ell I}(s)$ determined from the phase shift only (elastic regime)

$$t_{\ell I}(s) = \frac{e^{\delta_{\ell I}(s)}\, \sin\delta_{\ell I}(s)}{\sigma(s)}$$

Effective range expansion at low p

$$\frac{1}{m_\pi}\mathrm{Re}\, t_{\ell I}(s) = p^{2\ell}\left(\boldsymbol{a_{\ell I}} + \boldsymbol{b_{\ell I}}\, p^2 + \frac{1}{2}\boldsymbol{c_{\ell I}} p^4 + \ldots\right)$$

Scattering length ($a_{\ell I}$) Slope parameter ($b_{\ell I}$) Shape parameter ($c_{\ell I}$)

Contributions to threshold parameters

		$O(p^2)$	$O(p^4)$		$O(p^6)$		
		pol.	l_i pol.	J	b_i pol.	$b_i\,J$	K
$\ell = 0 \Rightarrow S$ *wave*	a_S	X	X	X	X	X	X
$\frac{\text{Re}\, t_0(s)}{m_\pi} = \left(a_S + b_S\, p^2 + \frac{1}{2} c_S p^4 + \ldots\right)$	b_S	X	X	X	X	X	X
	c_S		X	X	X	X	X
$\ell = 1 \Rightarrow P$ *wave*	a_P	X	X	X	X	X	X
$\frac{\text{Re}\, t_1(s)}{m_\pi} = \left(a_P p^2 + b_P\, p^4 + \frac{1}{2} c_P p^6 + \ldots\right)$	b_P		X	X	X	X	X
	c_P			X	X	X	X
$\ell = 2 \Rightarrow D$ *wave*	a_D		X	X	X	X	X
$\frac{\text{Re}\, t_3(s)}{m_\pi} = \left(a_D p^4 + b_D\, p^6 + \frac{1}{2} c_D p^8 + \ldots\right)$	b_D			X	X	X	X
	c_D			X		X	X
$\ell = 3 \Rightarrow F$ *wave*	a_F			X	X	X	X
$\frac{\text{Re}\, t_4(s)}{m_\pi} = \left(a_F p^6 + b_F\, p^8 + \frac{1}{2} c_F p^{10} + \ldots\right)$	b_F			X		X	X
	c_F			X		X	X

Sum rules

We use the threshold parameters calculated in * using **Froissart-Gribov sum rules** for $\ell > 0$

$$a_{\ell I} = \frac{\sqrt{\pi}\,\Gamma(\ell+1)}{4m_\pi \Gamma(\ell+3/2)} \int_{4m_\pi^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F_{(I)}(s, 4m_\pi^2)}{s^{\ell+1}}$$

$$b_{\ell I} = \frac{\sqrt{\pi}\,\Gamma(\ell+1)}{2m_\pi \Gamma(\ell+3/2)} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \left\{ \frac{4\mathrm{Im}\, F_{(I)}{}'_{\cos\theta}(s, 4m_\pi^2)}{(s-4m_\pi^2)s^{\ell+1}} - \frac{(\ell+1)\mathrm{Im}\, F_{(I)}(s, 4m_\pi^2)}{s^{\ell+2}} \right\}$$

(obtained by projecting a dispersion relation -or its derivative- over the ℓth partial wave in the t channel)

and **fast converging sum rules** for b_{S0}, b_{S2} and b_P

* R. Garcia-Martin, R. Kaminski, J. R. Pelaez, J. Ruiz de Elvira, F. J. Yndurain, Phys. Rev. **D83**, 074004 (2011).

In order to calculate the $c_{\ell I}$ **parameters**, we use the **Froissart-Gribov sum rule** for $\ell > 0$ for them

$$
\begin{aligned}
c_{\ell I} \;=\; & \frac{\sqrt{\pi}\,\Gamma(\ell+1)}{m_\pi\,\Gamma(\ell+3/2)} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \left\{ \frac{16\,\mathrm{Im}\,F_{(I)}{}''_{\cos\theta}(s,4m_\pi^2)}{(s-4m_\pi^2)^2 s^{\ell+1}} \right. \\
& - \left. 8(\ell+1)\frac{\mathrm{Im}\,F_{(I)}{}'_{\cos\theta}(s,4m_\pi^2)}{(s-4m_\pi^2)s^{\ell+2}} + \frac{\mathrm{Im}\,F_{(I)}(s,4m_\pi^2)}{s^{\ell+3}}\,\frac{(\ell+2)^2(\ell+1)}{\ell+3/2} \right\},
\end{aligned}
$$

and three additional **fast converging sum rules** for c_{S0}, c_{S2} and c_P

$$c_P = -\frac{14\,a_F}{3} + \frac{16}{3m_\pi}\int_{4m_\pi^2}^{\infty} ds' \left\{ \frac{\mathrm{Im}\, F_{I=0}(s)}{3s'^4} - \frac{\mathrm{Im}\, F_{I=1}(s)}{2s'^4} - \frac{5\mathrm{Im}\, F_{I=2}(s)}{6s'^4} + \left[\frac{\mathrm{Im}\, F_{I=1}(s)}{(s-4m_\pi^2)^4} - \frac{3a_P^2 m_\pi}{4\pi(s-4m_\pi^2)^{3/2}} \right] \right\}$$

$$c_{S2} = -6b_P - 10a_{D2} + \frac{8}{m_\pi}\int_{4m_\pi^2}^{\infty} \left\{ \frac{\mathrm{Im}\, F^{0+}(s)}{s^3} + \frac{1}{(s-4m_\pi^2)^{5/2}} \times \left[\frac{\mathrm{Im}\, F^{0+}(s)}{\sqrt{s-4m_\pi^2}} - \frac{2m_\pi a_{S2}^2}{\pi} - \frac{s-4m_\pi^2}{\pi}\left(\frac{m_\pi}{2}(2a_{S2}b_{S2} + a_{S2}^4) - \frac{a_{S2}^2}{4m_\pi} \right) \right] \right\}$$

$$c_{S0} = -2c_{S2} - 20a_{D2} - 10a_{D0} + \frac{12}{m_\pi}\int_{4m_\pi^2}^{\infty} \left\{ \frac{\mathrm{Im}\, F^{00}(s)}{s^3} + \frac{1}{(s-4m_\pi^2)^{5/2}} \times \left[\frac{\mathrm{Im}\, F^{00}(s)}{\sqrt{s-4m_\pi^2}} - \frac{4m_\pi(2a_{S2}^2 + a_{S0}^2)}{3\pi} - \frac{s-4m_\pi^2}{3\pi}\left(m_\pi(2a_{S2}b_{S2} + a_{S2}^4 + 2a_{S0}b_{S0} + a_{S0}^4) - \frac{2a_{S2}^2 + a_{S0}^2}{2m_\pi} \right) \right] \right\}$$

Threshold limit of the second derivative of a forward dispersion relation for $F_{I_s=1}$, F^{0+} and F^{00} $\left(F^{0+} = \frac{F_{I_s=2}}{2} + \frac{F_{I_s=1}}{2}, \quad F^{00} = 2\frac{F_{I_s=2}}{3} + \frac{F_{I_s=0}}{3}\right)$

Results

We fit the ChPT expressions for the threshold parameters.

In order to see how the series converge, we make

- One-loop fits
- Two-loop fits

One-loop fits

One-loop fits

	$O(p^2)$	$O(p^4)$	
	pol.	l_i pol.	J
a_{S0}, a_{S2}	x	x	x
b_{S0}, b_{S2}	x	x	x
c_{S0}, c_{S2}		x	x
a_P	x	x	x
b_P		x	x
c_P			x
a_{D0}, a_{D2}		x	x
b_{D0}, b_{D2}			x
c_{D0}, c_{D2}			x
a_F			x
b_F			x
c_F			x

- Four parameters:
  $$\bar{l}_i \propto l_i^r(\mu)|_{\mu=m_\pi}$$
- Only ten observables carry dependence on LECs
- Only five observables have $\mathcal{O}(p^2)$ as leading contribution

- $a_{S(0,2)}$, $b_{S(0,2)}$, a_P: observables for which the leading contribution is of $O(p^2)$
- $a_{D(0,2)}$: commonly used for the determination of l_1 and l_2

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
$a_{S(0,2)}$, $b_{S(0,2)}$, a_P	1.1±1.0	5.1±0.7	-1±8	7.1±0.7	0.23
$a_{D(0,2)}$	-1.75±0.22	5.91±0.10	—	—	0

Incompatible fits

If we include the 10 observables containing l_i, the incompatibility is even clearer

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
All	-2.06±0.14	5.97±0.07	-5±8	7.1±0.6	7.9
All, $f_\pi \leftrightarrow f_0$	-1.06±0.11	4.6±0.9	0±6	5.0±0.3	7.06

If one insists in using $O(p^4)$ for simplicity, one needs to sacrifice precision

-1.5±0.5	**5.2±0.7**	**-2±7**	**6.0±1.2**

Hence, a precise description calls for **higher order corrections**

Two-loop fits

Two-loop fits

	$O(p^2)$	$O(p^4)$		$O(p^6)$		
	pol.	l_i pol.	J	b_i pol.	$b_i\,J$	K
a_{S0}, a_{S2}	x	x	x	x	x	x
b_{S0}, b_{S2}	x	x	x	x	x	x
c_{S0}, c_{S2}		x	x	x	x	x
a_P	x	x	x	x	x	x
b_P		x	x	x	x	x
c_P			x	x	x	x
a_{D0}, a_{D2}		x	x	x	x	x
b_{D0}, b_{D2}			x	x	x	x
c_{D0}, c_{D2}			x		x	x
a_F			x	x	x	x
b_F			x		x	x
c_F			x		x	x

- Six parameters $\bar{b}_i$
- 18 observables
- Ten observables have $\mathcal{O}(p^4)$ contributions depending on l_i

Two-loop fits

- $a_{S,P,D}$, $b_{S,P}$, c_S: we fit the same 10 observables (those for which the $\mathcal{O}(p^4)$ contribution depends on the l_i)

	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$	$\bar{b}_5$	$\bar{b}_6$	$\chi^2/d.o.f.$
$a_{S,P,D}$, $b_{S,P}$, c_S	-14±4	14.6±1.2	-0.29±0.05	0.76±0.02	0.1±1.1	2.2±0.2	1.19

At two loops, the ten observables are well fitted

Two-loop fits

However, if we include in the fit

- All: 18 threshold parameters

	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$	$\bar{b}_5$	$\bar{b}_6$	$\chi^2/d.o.f.$
$a_{S,P,D}$, $b_{S,P}$, c_S	-14±4	14.6±1.2	-0.29±0.05	0.76±0.02	0.1±1.1	2.2±0.2	1.19
All	-2±3	14.2±1.0	-0.39±0.04	0.746±0.013	3.1±0.3	2.58±0.12	5.2

Not such a good fit

Hints to the need of **higher order corrections** in order to describe the threshold parameters at the current level of precision

Two-loop fits

We observe that the larger contribution to the χ^2 comes from c_P

- W/o c_P: All parameters except c_P

	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$	$\bar{b}_5$	$\bar{b}_6$	$\chi^2/d.o.f.$
$a_{S,P,D}$, $b_{S,P}$, c_S	-14±4	14.6±1.2	-0.29±0.05	0.76±0.02	0.1±1.1	2.2±0.2	1.19
All	-2±3	14.2±1.0	-0.39±0.04	0.746±0.013	3.1±0.3	2.58±0.12	5.2
W/o c_P	-6±3	15.9±1.1	-0.36±0.04	0.753±0.012	2.2±0.3	2.44±0.12	2.9

We repeat the fit without c_P replacing f_π by f_0 in $O(p^6)$ terms (higher order effect)

	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$	$\bar{b}_5$	$\bar{b}_6$	$\chi^2/d.o.f.$
W/o c_P	-6±3	15.9±1.1	-0.36±0.04	0.75±0.01	2.2±0.3	2.4±0.1	2.9
W/o c_P $f_\pi \leftrightarrow f_0$	-12±3	13.9±0.9	-0.30±0.04	0.73±0.01	1.0±0.3	1.9±0.1	1.04
Our estimate	**-7±6**	**14±2**	**-0.31±0.07**	**0.73±0.02**	**1.2±1.1**	**2.0±0.5**	

↑

weighted average with systematic errors to include both results

We repeat the fit without c_P replacing f_π by f_0 in $O(p^6)$ terms (higher order effect)

	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$	$\bar{b}_5$	$\bar{b}_6$	$\chi^2/d.o.f.$
W/o c_P	-6±3	15.9±1.1	-0.36±0.04	0.75±0.01	2.2±0.3	2.4±0.1	2.9
W/o c_P $f_\pi \leftrightarrow f_0$	-12±3	13.9±0.9	-0.30±0.04	0.73±0.01	1.0±0.3	1.9±0.1	1.04
Our estimate	**-7±6**	**14±2**	**-0.31±0.07**	**0.73±0.02**	**1.2±1.1**	**2.0±0.5**	
CGL *	-13±1	11.7±0.9	-0.33±0.12	0.74±0.03	3.6±1.7	2.4±0.2	

Results compatible with previous determinations

* G. Colangelo, J. Gasser, and H. Leutwyler, Nucl. Phys. B **603** (2001) 125.

Summary

Summary

We have calculated the threshold parameters c_ℓ by using a precise dispersive data anaylsis in sum rules.

We have shown results of one and two-loop fits:

- One loop (4 parameters, $\bar{l}_i$):

  - not enough to describe observables with precision

- Two loops (6 parameters, $\bar{b}_i$):

  - all observables except for c_P are well described
  - with parameters consistent with previous determinations
  - but, at least c_P, calls for even higher order corrections

Thank you!

Resulting observables:

	Avg. $O(p^4)$	Avg. $O(p^6)$	Sum rules
a_{S0}	0.213 ± 0.009	0.235 ± 0.015	0.220 ± 0.008
$a_{S2}(\times 10^2)$	-4.45 ± 0.3	-4.1 ± 0.4	-4.2 $\pm$ 0.4
$a_P(\times 10^3)$	38.6 ± 1.2	38.8 ± 0.9	38.1 ± 0.9
$a_{D0}(\times 10^4)$	15 ± 3	16.8 ± 0.6	17.8 ± 0.3
$a_{D2}(\times 10^4)$	1.3 ± 1.0	1.8 ± 0.3	1.85 ± 0.18
$a_F(\times 10^5)$	—	$\underline{4.6 \pm 0.5}$	5.65 ± 0.23
b_{S0}	$\underline{0.254 \pm 0.010}$	0.270 ± 0.008	0.278 ± 0.005
$b_{S2}(\times 10^2)$	-8.2 ± 0.5	-8.4 ± 0.3	-8.2 ± 0.4
$b_P(\times 10^3)$	$\underline{4.4 \pm 0.5}$	5.1 ± 0.2	5.37 ± 0.14
$b_{D0}(\times 10^4)$	—	-3.6 ± 0.4	-3.5 ± 0.2
$b_{D2}(\times 10^4)$	—	-3.1 ± 0.4	-3.3 ± 0.1
$b_F(\times 10^5)$	—	$\underline{-3.5 \pm 0.3}$	-4.06 ± 0.27
$c_{S0}(\times 10^2)$	2.3 ± 1.3	1.2 ± 0.7	0.45 ± 0.67
$c_{S2}(\times 10^2)$	3.4 ± 0.7	2.8 ± 0.14	2.80 ± 0.24
$c_P(\times 10^3)$	—	0.3 ± 0.2	1.39 ± 0.12
c_{D0}	—	$\underline{\underline{3.6 \pm 0.2}}$	4.4 ± 0.3
c_{D2}	—	$\underline{3.2 \pm 0.2}$	3.6 ± 0.2
$c_F(\times 10^5)$	—	$\underline{5.6 \pm 0.5}$	6.9 ± 0.4

One-loop fits

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
with LO	1.1±1.0	5.1±0.7	-1±8	7.1±0.7	0.23

- With LO: five observables with $\mathcal{O}(p^2)$ contribution

One-loop fits

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
with LO	1.1±1.0	5.1±0.7	-1±8	7.1±0.7	0.23
D-waves	-1.75±0.22	5.91±0.10	—	—	0

- With LO: five observables with $\mathcal{O}(p^2)$ contribution
- D-waves: l_1 and l_2 from a_{D0} and a_{D2}

One-loop fits

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
with LO	1.1±1.0	5.1±0.7	-1±8	7.1±0.7	0.23
D-waves	-1.75±0.22	5.91±0.10	—	—	0
Only c_S	-2.4±0.9	4.8±0.4	—	—	0

- With LO: five observables with $\mathcal{O}(p^2)$ contribution
- D-waves: l_1 and l_2 from a_{D0} and a_{D2}
- Only c_s: l_1 and l_2 from c_{S0} and c_{S2}

One-loop fits

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
with LO	1.1±1.0	5.1±0.7	-1±8	7.1±0.7	0.23
D-waves	-1.75±0.22	5.91±0.10	—	—	0
Only c_S	-2.4±0.9	4.8±0.4	—	—	0
All	-2.06±0.14	5.97±0.07	-5±8	7.1±0.6	7.9

- With LO: five observables with $\mathcal{O}(p^2)$ contribution
- D-waves: l_1 and l_2 from a_{D0} and a_{D2}
- Only c_s: l_1 and l_2 from c_{S0} and c_{S2}
- All: ten observables fitted

One-loop fits

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
with LO	1.1±1.0	5.1±0.7	-1±8	7.1±0.7	0.23
D-waves	-1.75±0.22	5.91±0.10	—	—	0
Only c_S	-2.4±0.9	4.8±0.4	—	—	0
All	-2.06±0.14	5.97±0.07	-5±8	7.1±0.6	7.9
All f_0	-1.06±0.11	4.6±0.9	0±6	5.0±0.3	7.06

- With LO: five observables with $\mathcal{O}(p^2)$ contribution
- D-waves: l_1 and l_2 from a_{D0} and a_{D2}
- Only c_s: l_1 and l_2 from c_{S0} and c_{S2}
- All: ten observables fitted
- All f_0: same, replacing f_π by f_0 in $\mathcal{O}(p^4)$ terms

One-loop fits

	$\bar{l}_1$	$\bar{l}_2$	$\bar{l}_3$	$\bar{l}_4$	$\chi^2/d.o.f.$
with LO	1.1$\pm$1.0	5.1$\pm$0.7	-1$\pm$8	7.1$\pm$0.7	0.23
D-waves	-1.75$\pm$0.22	5.91$\pm$0.10	—	—	0
Only c_S	-2.4$\pm$0.9	4.8$\pm$0.4	—	—	0
All	-2.06$\pm$0.14	5.97$\pm$0.07	-5$\pm$8	7.1$\pm$0.6	7.9
All f_0	-1.06$\pm$0.11	4.6$\pm$0.9	0$\pm$6	5.0$\pm$0.3	7.06
Our estimate	**-1.5$\pm$0.5**	**5.2$\pm$0.7**	**-2$\pm$7**	**6.0$\pm$1.2**	—