The Lambda parameter and strange quark mass in two-flavor QCD

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talk based on [arXiv:1205.5380]

in collaboration with

F.Knechtli, B.Leder, M.Marinkovic, S.Schaefer, R.Sommer, F.Virotta

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Introductory remarks

Toolbox: lattice regularisation of QCD, including $N_{\rm f} = 2$ dynamical quarks

1. Compute fundamental parameters of QCD, i.e.,
renormalization group invariant (RGI) quantities

$$\Lambda_{\rm QCD}^{(N_{\rm f})}\ ; \qquad M_i\,, \quad i = {\rm u, d, s}, \ldots$$

non-perturbatively $\rightsquigarrow$ $\Lambda_{\overline{\rm MS}}^{(2)}, \overline{m}_{\rm s}(2{\rm GeV})$

2. Good control over systematic errors, like
autocorrelations; finite volume, discretisation, . . . , chiral extrapol.
(Coordinated Lattice Simulations, CLS, effort)

3. Scale setting through physical quantities as f_π, $f_{\rm K}$, m_Ω, . . .
to convert dim.less numbers $(af_\pi, af_{\rm K}, \ldots)$ $\rightsquigarrow$ a, L, . . . in physical units

4. Compute further quantities of interest
decay constants, masses, form factors, . . . , ALPHAs B-physics program

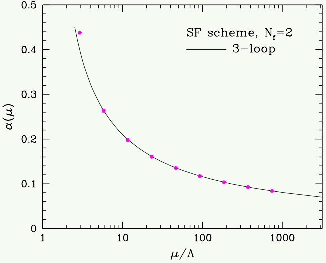
GENERAL SETUP

Non-perturbative Renormalization

Ingredient: Schrödinger functional as intermediate renormalization scheme

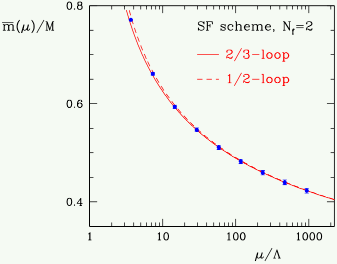
- massless, finite volume renorm. scheme in the continuum
- IR regulator on the lattice (Dirichlet b.c. in time) $\Rightarrow m = 0$ on the lattice
- NP definition of a running coupling $\Rightarrow \bar{g}^2(\mu)$, w/ box size $L = 1/\mu$
- $N_f = 2$: QCD running coupling [ALPHA'04] and mass [ALPHA'05] known through finite size scaling technique

NP running coupling:



$\hookrightarrow$ Λ-parameter, low energy scale $\mu \equiv 1/L_1$

NP running mass:



$\hookrightarrow$ RGI quark masses M_i

Dynamical fermion simulations

e.g.: O7 ensemble, $64^3 \times 128$, $m_\pi \sim 270\,\mathrm{MeV}$

Lattice framework:

- plaquette gauge action
- mass-degenerate doublet of non-perturbatively improved Wilson fermions

Our criteria:

- FV effects small by construction

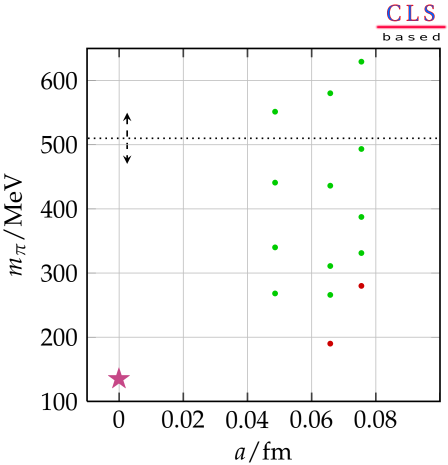
$$Lm_\pi \geq 4.0$$

- data for chiral extrapolation uses

$$m_\pi \lesssim 500\,\mathrm{MeV}$$

- three lattice spacings

$$< 0.08\,\mathrm{fm}$$



Goal: controlled extrapol. to physical point ★

Two strategies for chiral extrapolations

Setting the scale

Standard procedure, still room for improvements though

- calibrate lattice spacing a through dimensionful reference quantity Q:

$$a^{-1}[\mathrm{MeV}] = \frac{Q\big|_{\mathrm{exp}}[\mathrm{MeV}]}{\big[aQ\big]_{\mathrm{latt}}}\,, \quad Q \in \{f_\pi, f_\mathrm{K}, m_\mathrm{N}, m_\pi, \ldots\}$$

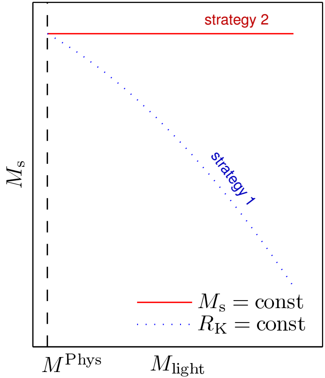
  choose well behaved quantity Q according to

  - experimentally available input
  - reasonable signal-to-noise ratio
  - well-controlled and understood chiral behaviour
  - mild cut-off effects
  - ...

Our choice: kaon decay constant f_K

- milder chiral extrapolation compared to f_π
- better control over systematic errors (2 strategies)

Two chiral extrapolations



Strategy 1

fix ratio:

$$\frac{m_K^2(\kappa_1, h(\kappa_1))}{f_K^2(\kappa_1, h(\kappa_1))} = R_K \stackrel{!}{=} \frac{m_{K,phys}^2}{f_{K,phys}^2}$$

- trajectory with $m_K \approx m_{K,phys}$ and thus

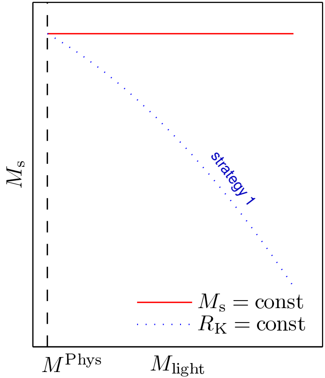
  $$\overline{m}_s + \overline{m}_{light} \approx \text{const} + O(\overline{m}^2)$$

- $\kappa_3 \equiv h(\kappa_1)$ determined by interpolation
- PQ-SU(3) ChPT: [Sharpe'97]

  $$f_K(\kappa_1, h(\kappa_1)) \longrightarrow f_{K,phys}$$

- systematic expansion in

  $$m_\pi^2, m_K^2 \leq m_{K,phys}^2$$



Strategy 2

fix strange quark's PCAC mass:

$$am_{34}(\kappa_1, s(\kappa_1,\mu)) \overset{!}{=} \mu$$

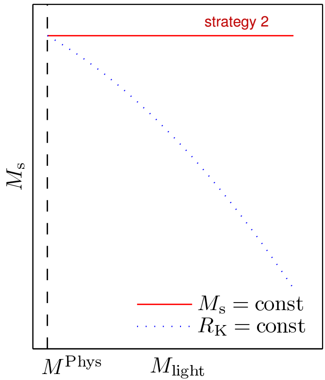
- μ fixed for 2 values (independent of κ_1)
- $\kappa_3 \equiv s(\kappa_1,\mu)$ determined by interpolation
- SU(2) ChPT: [Roessl:1999; AlltonEtAl:2008]

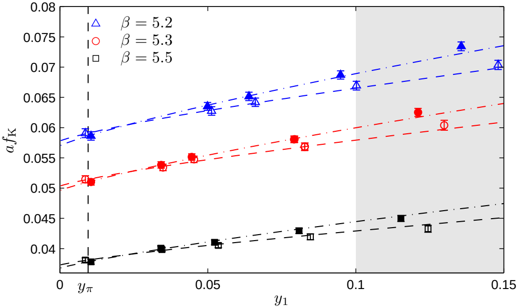
$$af_{\mathrm{K}}(\kappa_1, s(\kappa_1,\mu)) \longrightarrow p(\mu)$$

$$[am_{\mathrm{K}}]^2(\kappa_1, s(\kappa_1,\mu)) \longrightarrow q(\mu)$$

- solve numerically for μ_{s}:

$$\left.\frac{q(\mu)}{p(\mu)^2}\right|_{\mu=\mu_{\mathrm{s}}} \equiv \frac{m^2_{\mathrm{K,phys}}}{f^2_{\mathrm{K,phys}}}$$





- global fit
- expansion variable:

$$y_1 \equiv \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1, \kappa_3)} \qquad \leq 0.1$$

- ChPT to 1-loop, i.e., LEC or combinations thereof multiply $(y_1 - y_\pi)$
- $O(y^2)$ small

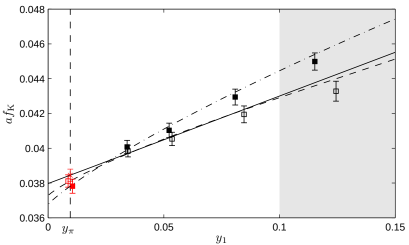
$f_{K,phys} = 155\,\text{MeV}$ [FLAG'11] (isospin symmetric limit & QED effects removed)

Strategy 1: △○□

- NLO partially quenched SU(3) ChPT
- $\kappa_3 = h(\kappa_1)$
- $a = [af_K]/f_{K,phys}$

Strategy 2: ▲●■

- NLO SU(2) ChPT
- $\kappa_3 = s(\kappa_1, \mu_s)$
- $a = p(\mu_s)/f_{K,phys}$



- cut at $y_1 = 0.1$

$$y_1 \equiv \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1)}$$

- global fit:
  α_4, α_f independent of β

check systematic from chiral extrapolation:

- chiral vs. linear fit (see above)
- |Strategy 1 - Strategy 2|

LEC's:

$$\alpha_4 = 0.57(12)\,, \qquad \alpha_f = 1.13(8)$$

taking all systematics into account:

β	af_K	a[fm]
5.2	0.0593(7)(6)	0.0755(9)(7)
5.3	0.0517(6)(6)	0.0658(7)(7)
5.5	0.0382(4)(3)	0.0486(4)(5)

RESULTS

Λ, M_{s}

Master formula: [ALPHA'05]

$$\frac{\Lambda^{(2)}_{\overline{\mathrm{MS}}}}{f_\mathrm{K}} = \frac{1}{[f_\mathrm{K} L_1]_\mathrm{cont}} \times [L_1 \Lambda^{(2)}_\mathrm{SF}]_\mathrm{cont} \times \frac{\Lambda^{(2)}_{\overline{\mathrm{MS}}}}{\Lambda^{(2)}_\mathrm{SF}}$$

Renorm. scale set through SF coupling:

- $\bar{g}^2(L_1) = 4.484, \qquad \mu_1 = 1/L_1$

Exact relation between schemes:

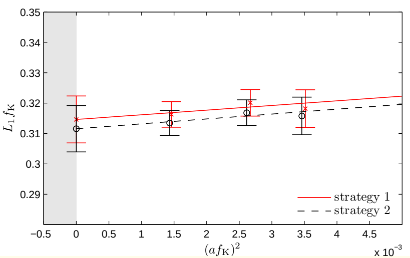
- $\Lambda^{(2)}_{\overline{\mathrm{MS}}} / \Lambda^{(2)}_\mathrm{SF} = 2.382035(3)$

Non-perturbative running coupling:

- $L_1 \Lambda^{(2)}_\mathrm{SF} = 0.264(15)$

Missing piece:

- $[f_\mathrm{K} L_1]_\mathrm{cont}$



The Λ parameter of $N_f = 2$ QCD

Master formula: [ALPHA'05]

$$\frac{\Lambda_{\overline{MS}}^{(2)}}{f_K} = \frac{1}{[f_K L_1]_{cont}} \times [L_1 \Lambda_{SF}^{(2)}]_{cont} \times \frac{\Lambda_{\overline{MS}}^{(2)}}{\Lambda_{SF}^{(2)}}$$

using result from strategy 1

$$L_1 f_K = 0.315(8)(2)$$

$$\Downarrow$$

$$\Lambda_{SF}^{(2)} / f_K = 0.84(6)$$

$$\Downarrow$$

$$\Lambda_{\overline{MS}}^{(2)} = 310(20)\,\text{MeV}$$

Renorm. scale set through SF coupling:

- $\bar{g}^2(L_1) = 4.484, \qquad \mu_1 = 1/L_1$

Exact relation between schemes:

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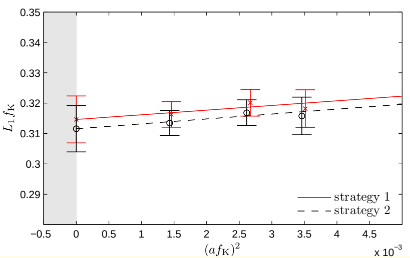

The strange quark mass in $N_\mathrm{f} = 2$ QCD

RGI strange quark mass:

$$M_\mathrm{s} = Z_\mathrm{M} m_\mathrm{s} = \frac{M}{\overline{m}(\mu_1)} \times \overline{m}_\mathrm{s}(\mu_1)$$
$$= \frac{M}{\overline{m}(\mu_1)} \times \frac{Z_\mathrm{A}}{Z_\mathrm{P}(\mu_1)} \times m_\mathrm{s}$$

Renorm. scale set through SF coupling:

- $\overline{g}^2(L_1) = 4.484, \qquad \mu_1 = 1/L_1$

Non-perturbative running mass: [ALPHA'05]

- $M/\overline{m}(\mu_1) = 1.308(16)$

The strange quark mass in $N_\mathrm{f} = 2$ QCD

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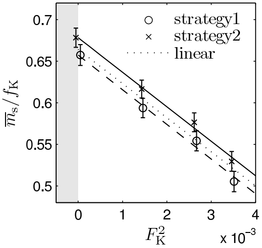


Strategy 1:

- only combination $\overline{m}_\mathrm{s} + \widehat{\overline{m}}$ directly accessible
- remove average light quark mass $\widehat{\overline{m}}$
  $\rightsquigarrow$ additional systematic uncertainty
- combination of LEC α_4 & α_6 from constrained global fit

Strategy 2: (*conceptually preferred*)

- no additional LEC's involved
- $\overline{m}_\mathrm{s}/f_\mathrm{K} = 0.678(12)(5)$

$$M_\mathrm{s} = 138(3)(1)\,\mathrm{MeV}\,, \qquad \overline{m}_\mathrm{s}^{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) = 102(3)(1)\,\mathrm{MeV}$$

Summary

- *complete analysis of CLS based* $N_f = 2$ *ensembles*; $a = (0.05 - 0.08)\,\mathrm{fm}$
- Conservative error estimates through autocorrelation analysis
- Scale setting with f_K
  - Two strategies for chiral extrapolation in agreement
  - simple linear extrapolation also agrees within errors
  - small cut-off effects

  $\rightsquigarrow$ control over systematic effects in scale setting
- Main results

$$\Lambda_{\overline{\mathrm{MS}}}^{(2)} = 310(20)\,\mathrm{MeV}\,, \qquad M_s = 138(3)(1)\,\mathrm{MeV}$$

$$\overline{m}_s^{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) = 102(3)(1)\,\mathrm{MeV}$$

  - controlled reduction of statistical & systematic error achieved (since 2005)
  - consistent with scale setting through r_0 (not coverred)

Outlook

- unclear situation

N_f	$\Lambda_{\overline{MS}}$	experiment	theory
0	$238(19)$ MeV	m_K, $K \to \mu\nu_\mu$, $K \to \pi\mu\nu_\mu$	lattice gauge theory [ALPHA'93]
2	$310(20)$ MeV	m_K, $K \to \mu\nu_\mu$, $K \to \pi\mu\nu_\mu$	this work
5	$212(12)$ MeV	world average	perturb. theory [Bethke'11]

$\rightsquigarrow$ remove remaining systematic uncertainty due to quenching $\rightsquigarrow N_f = 3, 4$

- ? TODO ?
  - other low energy constants
  - $\widehat{m}/m_s$
  - ...

Thank you for your attention!

...and many thanks to my colleagues

F.Knechtli, B.Leder, M.Marinkovic, S.Schaefer, R.Sommer, F.Virotta

Backup slides

Autocorrelations in HMC generated data

...grow towards the continuum limit

- autocorrel. depends on observable

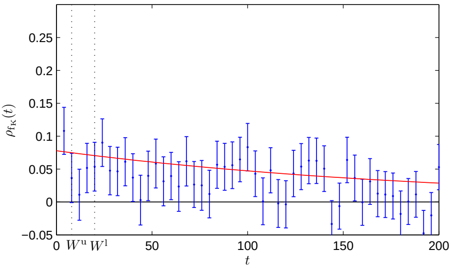
$$\overline{F} = \frac{1}{N}\sum_{i=1}^{N} F_i$$

- autocorrelation function

$$\Gamma_F(t) = \langle (F_t - \overline{F})(F_0 - \overline{F}) \rangle$$

- integrated autocorrelation time

$$\tau_{\rm int} = \frac{1}{2}\sum_{t=1}^{\infty} \rho_F(t) \ ,$$
$$\rho_F(t) = \Gamma_F(t)/\Gamma_F(0)$$



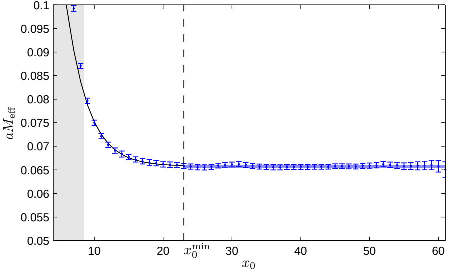
$$\tau_{\rm exp}(\beta) = 200\,\frac{c_\tau}{R_{\rm act}}\,{\rm e}^{7[\beta-5.3]}$$

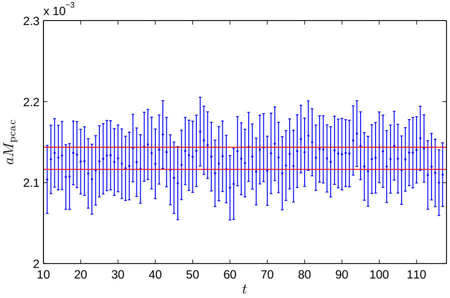
- $\tau_{\rm exp}$ estimated from $\beta = 5.3$ and quenched scaling [Schaefer,Sommer,Virotta'10]
- $R_{\rm act}$ – fraction of active links (DD-HMC: ∼30%; MP-HMC: 100%)
- $c_{\tau=2} = 1$; $c_{\tau=0.5} = 2$

Exponential tail accounts for ∼ 50% of the total uncertainty !

Masses and decay constants

A handle on excited states $\left(128 \times 64^3 \text{ lattice}, a = 0.045\text{fm}, m_\pi = 268\text{MeV}\right)$





Taking $\tau_{\rm exp}$ into account:

- $1^{\rm st}$ step: double exponential fit to $\{c_1, c_2, m\}$ with $m' = m + 2m_\pi$

$$F(x_0) = c_1\left(\mathrm{e}^{-mx_0} + \mathrm{e}^{-m(T-x_0)}\right) + c_2\left(\mathrm{e}^{-m'x_0} + \mathrm{e}^{-m'(T-x_0)}\right)$$

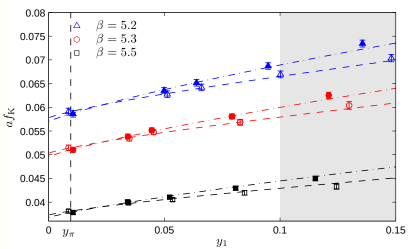
+ criterion: stat.error$(aM_{\rm eff}) \leq 4 \times 1^{\rm st}$ excited state contribution $\Rightarrow x_0^{\rm min}$

- $2^{\rm nd}$ step: single exponential fit $(c_2 \equiv 0)$

$c_2 \approx 0$ well justified theoretically; see talk by M.Golterman

Results for chiral extrapolations

Strategy 1



- strat.1: △○□ strat.2: ▲●■
- cut at $y_1 = 0.1$

$$y_1 \equiv \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1, h(\kappa_1))}$$

- constrained global fit: α_4 independent of a

partially quenched ChPT:

$$f_K(\kappa_1, h(\kappa_1)) = f_{K,\text{phys}} \left[1 + \overline{L}_K(y_1, y_K) + (\alpha_4 - \tfrac{1}{4})(y_1 - y_\pi) + O(y^2)\right],$$

$$\overline{L}_K(y_1, y_K) = L_K(y_1, y_K) - L_K(y_\pi, y_K),$$

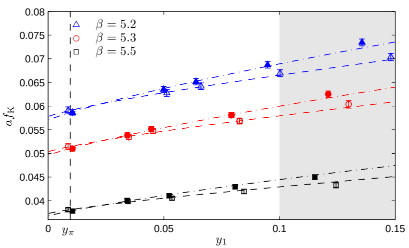
$$L_K(y_1, y_K) = -\tfrac{1}{2} y_1 \log(y_1) - \tfrac{1}{8} y_1 \log(2y_K/y_1 - 1).$$

$y_3 \equiv y_1(\pi \to K)$ does not appear, since $y_3 = 2y_K - y_1 + O(y^2)$

combination of chiral logs overall much smaller with less curvature compared to L_π

Results for chiral extrapolations

Strategy 2



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- cut at $y_1 = 0.1$

$$y_1 \equiv \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1, s(\kappa_1,\mu))}$$

- constrained global fit: α_f, α_m independent of a

SU(2) ChPT:

$$af_K(\kappa_1, s(\kappa_1,\mu)) = p(\mu)\left[1 - \tfrac{3}{8}[y_1\log(y_1) - y_\pi\log(y_\pi)] + \alpha_f(\mu)(y_1 - y_\pi) + O(y_1^2)\right],$$

$$a^2m_K^2(\kappa_1, s(\kappa_1,\mu)) = q(\mu)\left[1 + \alpha_m(\mu)(y_1 - y_\pi) + O(y_1^2)\right]$$

for three different fixed μ

$$\frac{q(\mu_s)}{p(\mu_s)^2} = \frac{m_{K,phys}^2}{f_{K,phys}^2} \quad \Rightarrow \quad a = \frac{p(\mu_s)}{f_{K,phys}}$$

The strange quark mass in $N_\mathrm{f} = 2$ QCD

RGI strange quark mass:

$$M_\mathrm{s} = Z_\mathrm{M} m_\mathrm{s} = \frac{M}{\overline{m}(\mu_1)} \times \overline{m}_\mathrm{s}(\mu_1) = \frac{M}{\overline{m}(\mu_1)} \times \frac{Z_\mathrm{A}}{Z_\mathrm{P}(\mu_1)} \times m_\mathrm{s}$$

Renorm. scale set through SF coupling:

- $\overline{g}^2(L_1) = 4.484, \qquad \mu_1 = 1/L_1$

Non-perturbative running mass: [ALPHA'05]

- $M/\overline{m}(\mu_1) = 1.308(16)$

Strategy 1:

$$\frac{2m_{13}(\kappa_1, h(\kappa_1))}{Z_\mathrm{P} f_\mathrm{K}(\kappa_1, h(\kappa_1))} = \frac{\overline{m}_\mathrm{s} + \widehat{\overline{m}}}{f_\mathrm{K,phys}} \left[1 + \overline{L}_\mathrm{m}(y_1, y_\mathrm{K}) + (\alpha_{4,6} - \tfrac{1}{4})(y_1 - y_\pi) + \mathrm{O}(y_1^2)\right],$$

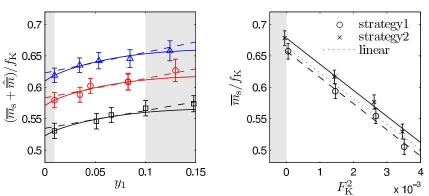
$$\overline{L}_\mathrm{m}(y_1, y_\mathrm{K}) = L_\mathrm{m}(y_1, y_\mathrm{K}) - L_\mathrm{m}(y_\pi, y_\mathrm{K}), \qquad \alpha_{4,6} = 3\alpha_4 - 4\alpha_6,$$

$$L_\mathrm{m}(y_1, y_\mathrm{K}) = -(y_\mathrm{K} - \tfrac{3}{8} y_1) \log(2y_\mathrm{K}/y_1 - 1) - y_\mathrm{K} \log(y_1)$$

Strategy 2: (*conceptually preferred*)

$$\frac{M_\mathrm{s}}{f_\mathrm{K,phys}} = \frac{M}{\overline{m}(L_1)} \times \frac{1}{Z_\mathrm{P}(L_1)} \times [1 - \widetilde{b}_\mathrm{P} \mu_\mathrm{s}] \frac{\mu_\mathrm{s}}{F^\mathrm{bare}_\mathrm{K,phys}}$$

The strange quark mass in $N_\mathrm{f} = 2$ QCD



$\overline{m}_\mathrm{s} / f_\mathrm{K}$ = 0.678(12)(5)

Strategy 1:

- use $y_1 = m_\pi^2(\kappa_1)/8\pi^2 f_\mathrm{K}^2(\kappa_1) < 0.1$
- remove avg. light quark contribution $\widehat{\overline{m}}$ by correction factor ρ: $M_\mathrm{s} = (M_\mathrm{s} + \widehat{M})(1-\rho)$

  LOχPT : $\quad \rho = \widehat{M}/(M_\mathrm{s} + \widehat{M}) \approx \frac{m_\pi^2}{2m_\mathrm{K}^2}$
- LEC $\alpha_{4,6}$ from constrained global fit

Strategy 2:

- neglect tiny $\mathrm{O}(a)$ impr. term, $(\bar{b}_\mathrm{A} - \bar{b}_\mathrm{P})am_\mathrm{sea}$

Strategy 1:

$$\frac{2m_{13}(\kappa_1, h(\kappa_1))}{Z_\mathrm{P} f_\mathrm{K}(\kappa_1, h(\kappa_1))} = \frac{\overline{m}_\mathrm{s} + \widehat{\overline{m}}}{f_\mathrm{K,phys}} \left[1 + \overline{L}_\mathrm{m}(y_1, y_\mathrm{K}) + (\alpha_{4,6} - \tfrac{1}{4})(y_1 - y_\pi) + \mathrm{O}(y_1^2)\right],$$

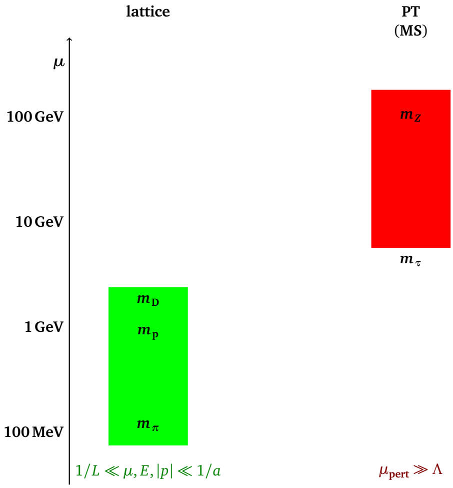
$$M_\mathrm{s} = 138(3)(1)\,\mathrm{MeV} \qquad \overline{m}_\mathrm{s}^{\overline{\mathrm{MS}}}(\mu = 2\mathrm{GeV}) = 102(3)(1)\,\mathrm{MeV}$$

Strategy 2: (*conceptually preferred*)

$$\frac{M_\mathrm{s}}{f_\mathrm{K,phys}} = \frac{M}{\overline{m}(L_1)} \times \frac{1}{Z_\mathrm{P}(L_1)} \times [1 - \widetilde{b}_\mathrm{P}\mu_\mathrm{s}] \frac{\mu_\mathrm{s}}{F_\mathrm{K,phys}^\mathrm{bare}}$$

one more important ingredient:

How to connect hadronic observables from low-energies to the widely used $\overline{\text{MS}}$-scheme?



Generic strategy

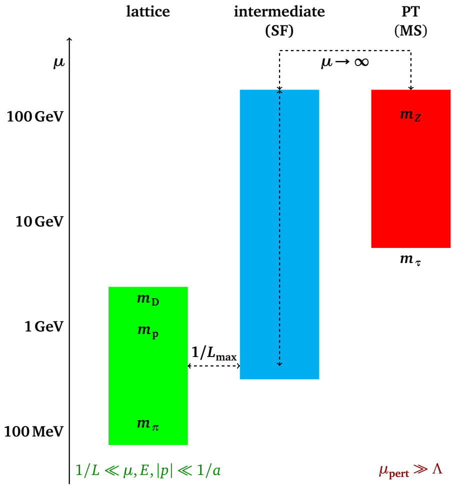
. . . to connect low- & high-energy regime NP'ly

one more important ingredient:

How to connect hadronic observables from low-energies to the widely used $\overline{\mathrm{MS}}$-scheme?

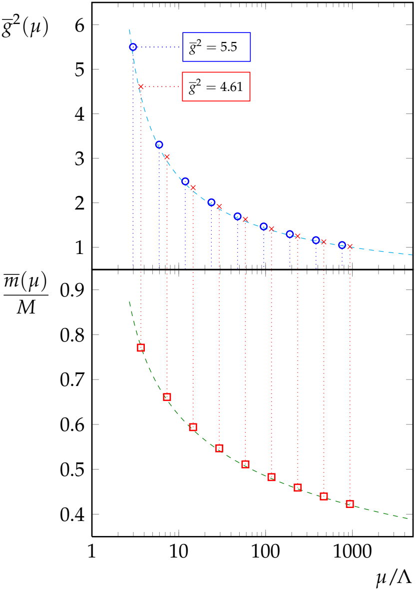
intermediate renorm. scheme
$\equiv$
Schrödinger functional
(finite volume, continuum scheme)

- RG scale evolution solved NP'ly
- thus continuum limit needs to be well controled (small cutoff effects)
- low-energy scale fixed by imposing $\bar{g}^2(L_{\max}) \equiv u_{\max}$



Scale dependence of QCD parameters

Running coupling and mass,



Renormalization group (RG) equations

1. coupling

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3(b_0 + b_1 \bar{g}^2 + \ldots)$$

2. mass

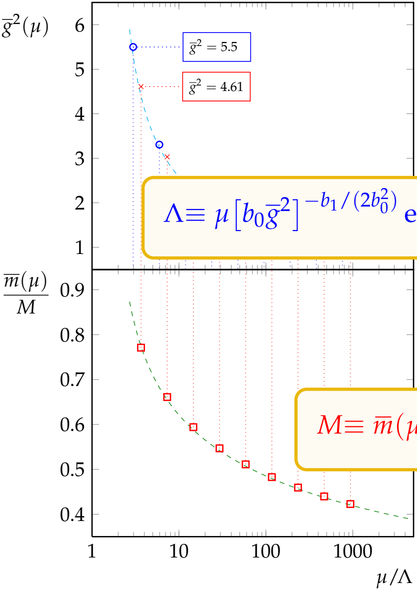
$$\frac{\mu}{\overline{m}} \frac{\partial \overline{m}}{\partial \mu} = \tau(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^2(d_0 + d_1 \bar{g}^2 + \ldots)$$

in a massless scheme, b_0, b_1, d_0 universal

Solution leads to *exact* equations in mass-independent scheme

Scale dependence of QCD parameters

Running coupling and mass, Renormalization Group Invariants (RGI)



Renormalization group (RG) equations

1. coupling

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3(b_0 + b_1\bar{g}^2 + \dots)$$

$$\Lambda \equiv \mu \left[b_0\bar{g}^2\right]^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}^2)} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

2. mass

$$\frac{\mu}{\overline{m}} \frac{\partial \overline{m}}{\partial \mu} = \tau(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^2(d_0 + d_1\bar{g}^2 + \dots)$$

$$M \equiv \overline{m}(\mu)\left[2b_0\bar{g}^2\right]^{-d_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}g \left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right\}$$

in a massless scheme, b_0, b_1, d_0 universal

Solution leads to *exact* equations in mass-independent scheme