Dispersive analysis of the scalar form factor of the nucleon

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Bonn-Cologne Graduate School of Physics and Astronomy

Outline

1. Extracting the pion–nucleon σ term from πN scattering

2. Dispersion relations for the scalar pion and kaon form factors
   - Unitarity relation
   - Two-channel Muskhelishvili–Omnès problem

3. Dispersion relations for the scalar form factor of the nucleon
   - From Roy–Steiner equations to the scalar form factor
   - Results

The mass of the nucleon

Decomposition of the nucleon mass

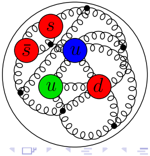
$$m = \frac{1}{\langle N|N\rangle}\langle N|\underbrace{\frac{\beta_{\text{QCD}}}{2g}F^a_{\mu\nu}F^{\mu\nu}_a}_{\text{trace anomaly}} + \underbrace{m_{\text{u}}\bar{u}u + m_{\text{d}}\bar{d}d + m_{\text{s}}\bar{s}s + \cdots}_{\text{Higgs}}|N\rangle$$

- Mass largely generated by gluon field energy via the **trace anomaly of the QCD energy-momentum tensor** $\theta^\mu_\mu \neq 0$
- $m_{\text{u}}, m_{\text{d}} \ll m$, but scheme dependent

$$\sigma_{\pi N} = \frac{1}{\langle N|N\rangle}\langle N|\hat{m}(\bar{u}u + \bar{d}d)|N\rangle \sim 50\,\text{MeV}$$

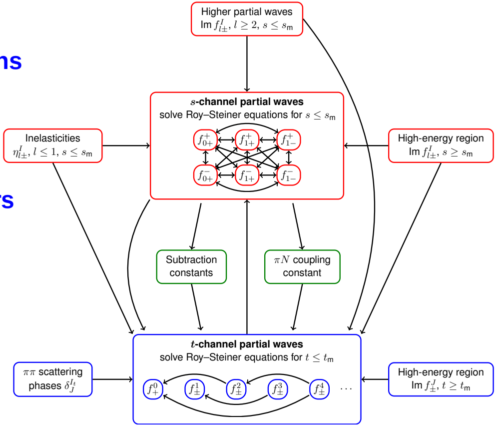
$\hookrightarrow$ measures light-quark contribution to the nucleon mass, how large precisely?

m_{u}	$(2.2 \pm 0.2)\,\text{MeV}$	FLAG 2010
m_{d}	$(4.7 \pm 0.2)\,\text{MeV}$	
m	$940\,\text{MeV}$	



Roy–Steiner equations and the scalar form factor

- Previous talk: **Roy–Steiner equations** for pion–nucleon scattering
- Solution of t-channel equations
  - $\hookrightarrow$ **spectral function of form factors**
  - $\hookrightarrow$ vector form factors (P-waves)
  - $\hookrightarrow$ scalar form factor (S-wave)



- Unitarity relation for $f^0_+(t)$: need $\pi\pi$ and $\bar{K}K$ states
  - $\leftrightarrow$ $f_0(980)$ resonance
  - $\hookrightarrow$ RS framework for a **fully consistent inclusion of $\bar{K}K$ intermediate states**
- Essential for **σ-term extraction**

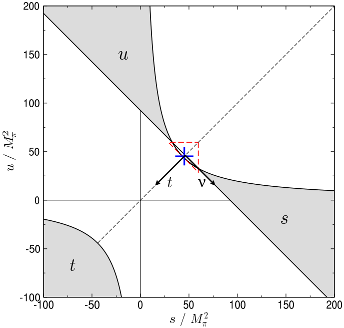
Extraction of $\sigma_{\pi N}$ from πN scattering: how-to

- **Scalar form factor** of the nucleon

$$\sigma(t) = \frac{1}{2m}\langle N(p')|\hat{m}(\bar{u}u+\bar{d}d)|N(p)\rangle \qquad t=(p'-p)^2 \qquad \sigma_{\pi N}=\sigma(0)$$

- **Low-energy theorem** Cheng, Dashen 1971

$$F_\pi^2\bar{D}^+(\nu=0,t=2M_\pi^2)=\sigma(2M_\pi^2)+\Delta_R$$



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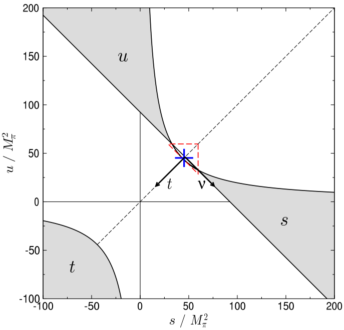

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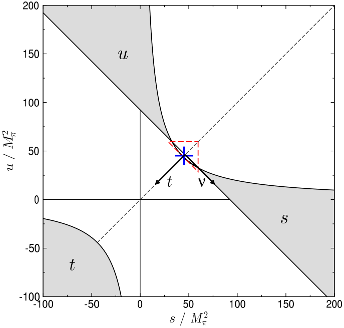

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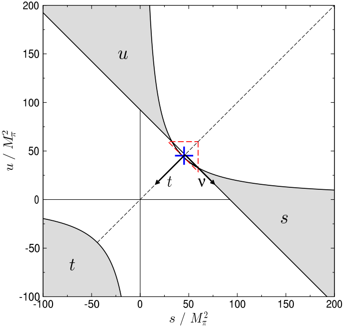
- Remainder $|\Delta_R| \lesssim 2\,\text{MeV}$ small Bernard, Kaiser, Meißner 1996



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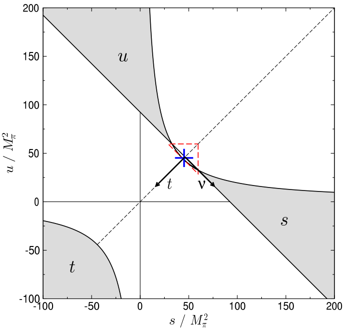
- Remainder $|\Delta_R| \lesssim 2\,\text{MeV}$ small Bernard, Kaiser, Meißner 1996
- **Dispersive approach** Gasser, Leutwyler, Sainio 1991

$$\Delta_D - \Delta_\sigma = (-3.3\pm 0.2)\,\text{MeV}$$

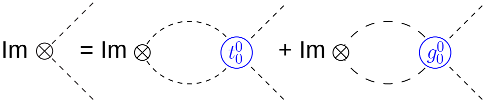
but error only covers $\pi\pi$ phase shifts

- **This talk**: update determination of Δ_D and Δ_σ
  - $\bar{K}K$ intermediate states
  - Dependence on πN parameters



Unitarity relation for the scalar pion and kaon form factors

- Consider first **meson form factors** $\hookrightarrow$ needed as input for the nucleon case



$$\mathrm{Im}\,F_\pi^S(t) = \sigma_t^\pi \big(t_0^0(t)\big)^* F_\pi^S(t) + \frac{2}{\sqrt{3}}\sigma_t^K \big(g_0^0(t)\big)^* F_K^S(t)$$

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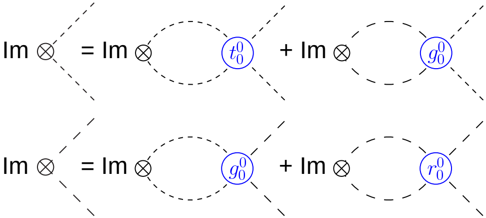

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$$\mathrm{Im}\,F_K^S(t) = \frac{\sqrt{3}}{2}\sigma_t^\pi \big(g_0^0(t)\big)^* F_\pi^S(t) + \sigma_t^K \big(r_0^0(t)\big)^* F_K^S(t)$$

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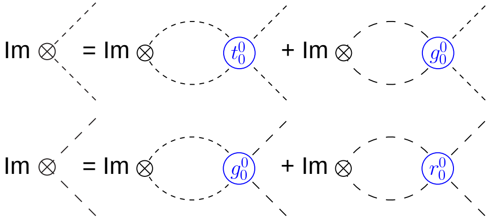

$$\text{Im}\,\boldsymbol{F}^S(t) = \big(T(t)\big)^* \Sigma(t) \boldsymbol{F}^S(t) \qquad \boldsymbol{F}^S(t) = \begin{pmatrix} F_\pi^S(t) \\ \frac{2}{\sqrt{3}} F_K^S(t) \end{pmatrix}$$

- Unitarity in the $\pi\pi/\bar{K}K$ system

$$T(t) = \begin{pmatrix} \frac{\eta_0^0(t) e^{2i\delta_0^0(t)} - 1}{2i\sigma_t^\pi} & |g(t)| e^{i\psi_0^0(t)} \\ |g(t)| e^{i\psi_0^0(t)} & \frac{\eta_0^0(t) e^{2i(\psi_0^0(t) - \delta_0^0(t))} - 1}{2i\sigma_t^K} \end{pmatrix} \qquad \Sigma(t) = \text{diag}\big(\sigma_t^\pi, \sigma_t^K\big)$$

$\hookrightarrow$ Two **phase shifts** δ_0^0, ψ_0^0, one **inelasticity parameter** $\eta_0^0 = \sqrt{1 - 4\sigma_t^\pi \sigma_t^K |g(t)|^2}$

Omnès matrix

Two-channel Muskhelishvili–Omnès problem

$$\operatorname{Im}\boldsymbol{\Omega}(t) = \big(T(t)\big)^{*}\Sigma(t)\boldsymbol{\Omega}(t)$$

- Two linearly independent solutions $\boldsymbol{\Omega}_1$, $\boldsymbol{\Omega}_2$ Muskhelishvili 1953
- In general **no analytical solution** for the Omnès matrix

$$\Omega(t) = \big\{\boldsymbol{\Omega}_1(t), \boldsymbol{\Omega}_2(t)\big\}$$

but for its determinant Moussallam 2000

$$\det\Omega(t) = \exp\left\{\frac{t}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}t' \frac{\psi_0^0(t')}{t'(t'-t)}\right\}$$

- Discretization $\Rightarrow$ matrix equation for $\Omega(t)$ Moussallam 2000
- Choose normalization $\Omega(0) = \mathbb{1}$

Scalar pion and kaon form factors

- Form factors obey the defining property of the Omnès matrix (no left-hand cut)

$$\boldsymbol{F}^S(t) = \alpha \boldsymbol{\Omega}_1(t) + \beta \boldsymbol{\Omega}_2(t)$$

- Fix normalization in terms of $\boldsymbol{F}^S(0)$ Donoghue, Gasser, Leutwyler 1990

$$F_\pi^S(t) = F_\pi^S(0)\Omega_{11}(t) + \frac{2}{\sqrt{3}} F_K^S(0)\Omega_{12}(t)$$

$$F_K^S(t) = \frac{\sqrt{3}}{2} F_\pi^S(0)\Omega_{21}(t) + F_K^S(0)\Omega_{22}(t)$$

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$$F^S_\pi(t) = F^S_\pi(0)\Omega_{11}(t) + \frac{2}{\sqrt{3}} F^S_K(0)\Omega_{12}(t)$$

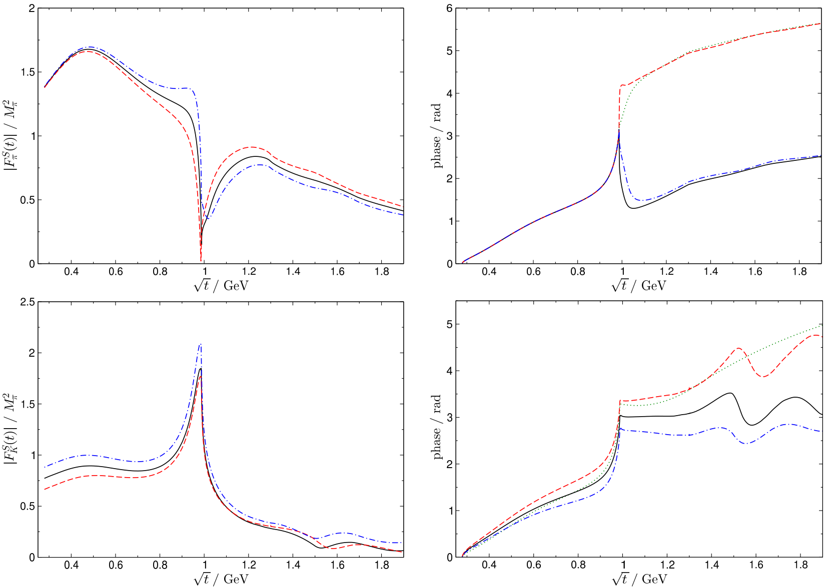
$$F^S_K(t) = \frac{\sqrt{3}}{2} F^S_\pi(0)\Omega_{21}(t) + F^S_K(0)\Omega_{22}(t)$$

- ChPT at $\mathcal{O}(p^4)$ with LECs from FLAG 2010

$$F^S_\pi(0) = (0.984 \pm 0.006) M_\pi^2 \qquad F^S_K(0) = (0.4 \ldots 0.6) M_\pi^2$$

- Further input
  - $\pi\pi$ phase shift δ^0_0 and inelasticity η^0_0 Caprini, Colangelo, Leutwyler (in preparation)
  - $|g(t)|$ from RS analysis of πK scattering Büttiker, Descotes-Genon, Moussallam 2004
  - $\pi\pi \to \bar{K}K$ phase shift ψ^0_0 from PWA Cohen et al. 1980, Etkin et al. 1982

Scalar pion and kaon form factors: results



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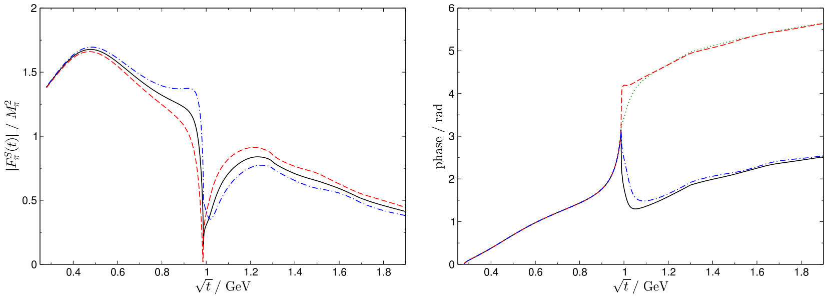
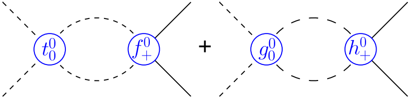


- $F_\pi^S(t)$ around 2-kaon threshold very sensitive to $F_K^S(0)$

  $\hookrightarrow$ two-channel nature of the problem
- Effective single-channel description in terms of the phase of $F_\pi^S(t)$ only possible for sufficiently large $F_K^S(0)$

Back to πN: unitarity relations



$$\mathrm{Im}\, f_+^0(t) = \sigma_t^\pi \left(t_0^0(t)\right)^* f_+^0(t) + \frac{2}{\sqrt{3}} \sigma_t^K \left(g_0^0(t)\right)^* h_+^0(t)$$

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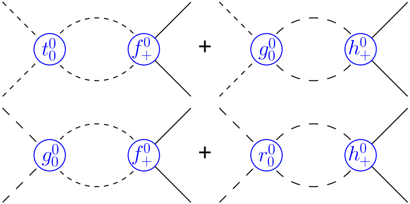

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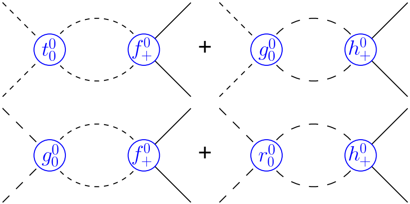

$$\mathrm{Im}\,\boldsymbol{f}(t) = \big(T(t)\big)^* \Sigma(t)\boldsymbol{f}(t) \qquad \boldsymbol{f}(t) = \begin{pmatrix} f_+^0(t) \\ \frac{2}{\sqrt{3}} h_+^0(t) \end{pmatrix}$$

- Same unitarity relation as before, but now **left-hand cut**

  $\hookrightarrow$ t-channel RS equations

- Closed system would require RS equations for KN scattering, but
  - **Bose symmetry** in $\pi\pi \Rightarrow$ only states with even/odd I and even/odd J
  - Nucleon pole $\rightarrow$ hyperon pole
  - Replace $M_\pi \rightarrow M_K$ in kernel functions

t-channel Roy–Steiner equations

Integral equation for the S-wave

$$\boldsymbol{f}(t) = \boldsymbol{\Delta}(t) + (\boldsymbol{a} + \boldsymbol{b}t)(t-4m^2) + \frac{t^2(t-4m^2)}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{\text{Im}\,\boldsymbol{f}(t')}{t'^2(t'-4m^2)(t'-t)}$$

- $\boldsymbol{\Delta}(t)$: Born terms, s-channel integrals, higher t-channel partial waves

  $\hookrightarrow$ **left-hand cut**

- $\boldsymbol{a}$, $\boldsymbol{b}$: **subthreshold parameters**

t-channel Roy–Steiner equations

Integral equation for the S-wave

$$f(t)=\Delta(t)+(a+bt)(t-4m^2)+\frac{t^2(t-4m^2)}{\pi}\int\limits_{4M_\pi^2}^{\infty}\mathrm{d}t'\frac{\mathrm{Im}\,f(t')}{t'^2(t'-4m^2)(t'-t)}$$

- $\Delta(t)$: Born terms, s-channel integrals, higher t-channel partial waves

  $\hookrightarrow$ **left-hand cut**

- a, b: **subthreshold parameters**
- Formal solution as in the single-channel case (now with Omnès matrix $\Omega(t)$)

$$f(t)=\Delta(t)+(t-4m^2)\Omega(t)(\mathbb{1}-t\,\dot{\Omega}(0))a+t(t-4m^2)\Omega(t)b$$
$$-\frac{t^2(t-4m^2)}{\pi}\Omega(t)\int\limits_{4M_\pi^2}^{t_\mathrm{m}}\mathrm{d}t'\frac{\mathrm{Im}\,\Omega^{-1}(t')\Delta(t')}{t'^2(t'-4m^2)(t'-t)}+\frac{t^2(t-4m^2)}{\pi}\Omega(t)\int\limits_{t_\mathrm{m}}^{\infty}\mathrm{d}t'\frac{\Omega^{-1}(t')\mathrm{Im}\,f(t')}{t'^2(t'-4m^2)(t'-t)}$$

- Main difficulty: need $\Omega(t)$ with **finite matching point**

Omnès matrix with finite matching point

Defining property

$$\begin{Bmatrix} \operatorname{Im}\Omega(t) = \big(T(t)\big)^{*}\Sigma(t)\Omega(t) \\ \operatorname{Im}\Omega(t) = 0 \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} 4M_\pi^2 \leq t \leq t_\text{m} \\ \text{otherwise} \end{Bmatrix}$$

- Again no analytical solution, only

$$\det\Omega(t) = \exp\left\{\frac{t}{\pi}\int_{4M_\pi^2}^{t_\text{m}} dt' \frac{\psi_0^0(t')}{t'(t'-t)}\right\}$$

- **Cusps** at the matching point

$$\Omega_{ij}(t) \sim |t_\text{m} - t|^{x_{ij}} \qquad \det\Omega(t) \sim |t_\text{m} - t|^{x} \qquad x = \frac{\psi_0^0(t_\text{m})}{\pi}$$

$$\begin{Bmatrix} x_{11} \\ x_{12} \end{Bmatrix} = \begin{Bmatrix} x_{21} \\ x_{22} \end{Bmatrix} = \frac{1}{2}\left\{x \pm \frac{1}{\pi}\arccos\big(\eta_0^0 \cos\pi(2y - x)\big)\right\} \qquad y = \frac{\delta_0^0(t_\text{m})}{\pi}$$

- Construct $\Omega(t)$ using infinite-matching-point solution and the known cusps

Input

- πN and KN **s-channel partial waves** Arndt et al. 2008, 1992
- $\boldsymbol{f}(t) = \mathbf{0}$ above t_m, **kinematical zero** at $t = 4m^2 \Rightarrow$ take $t_\mathrm{m} = 4m^2$

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- $\boldsymbol{f}(t) = \mathbf{0}$ above t_m, **kinematical zero** at $t = 4m^2 \Rightarrow$ take $t_\text{m} = 4m^2$
- Two-channel approximation breaks down around $\sqrt{t_0} = 1.3\,\text{GeV} \Rightarrow$ **4π channel**
- Phase shifts above t_0
  - "**RS1**": keep δ_0^0 and ψ_0^0 constant above t_0
  - "**RS2**": guide δ_0^0 and ψ_0^0 smoothly to $2\pi \Rightarrow$ meson form factors

  $\hookrightarrow$ assess uncertainty in phase-shift input

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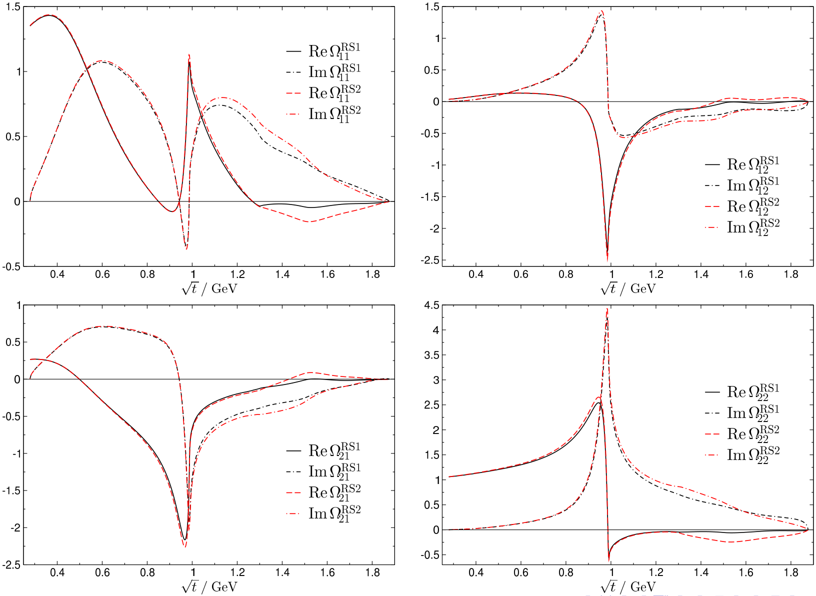
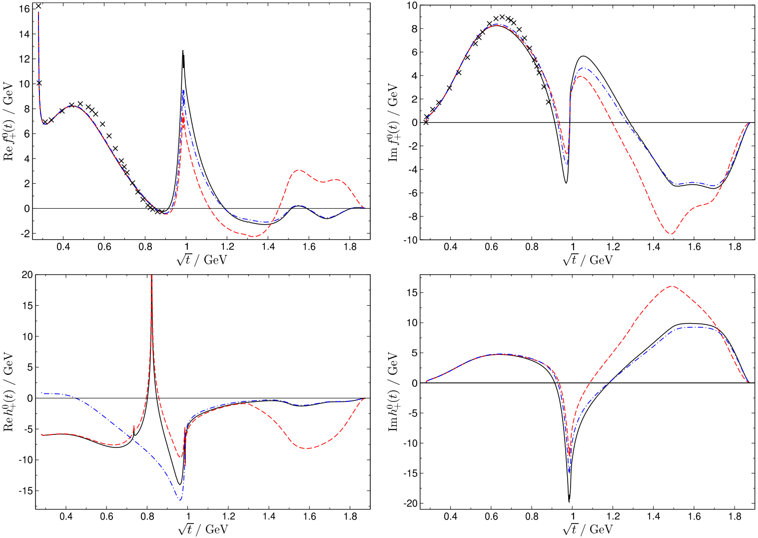
    $\hookrightarrow$ assess uncertainty in phase-shift input
- KH80 πN coupling and subthreshold parameters as reference point Höhler 1983
- Hyperon couplings from Jülich model 1989, KN subthreshold parameters neglected
  - "**RS3**": as **RS1**, but with $\Delta_2(t) = 0$

    $\hookrightarrow$ assess uncertainty in KN input

Results: Omnès matrix



Results: $f_+^0(t)$ and $h_+^0(t)$



solid ↔ RS1

dashed ↔ RS2

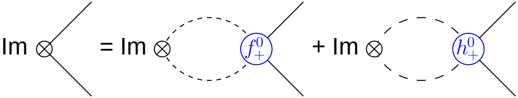
dot-dashed ↔ RS3

crosses ↔ KH80

↪ Main uncertainty due to **phase shifts** at higher energies

Dispersion relation for the scalar form factor of the nucleon

- Unitarity relation



$$\operatorname{Im}\sigma(t) = \frac{2}{4m^2 - t}\left\{\frac{3}{4}\sigma_t^\pi\left(F_\pi^S(t)\right)^* f_+^0(t) + \sigma_t^K\left(F_K^S(t)\right)^* h_+^0(t)\right\}$$

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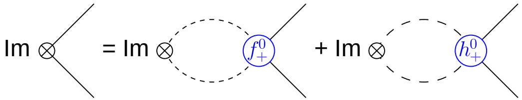

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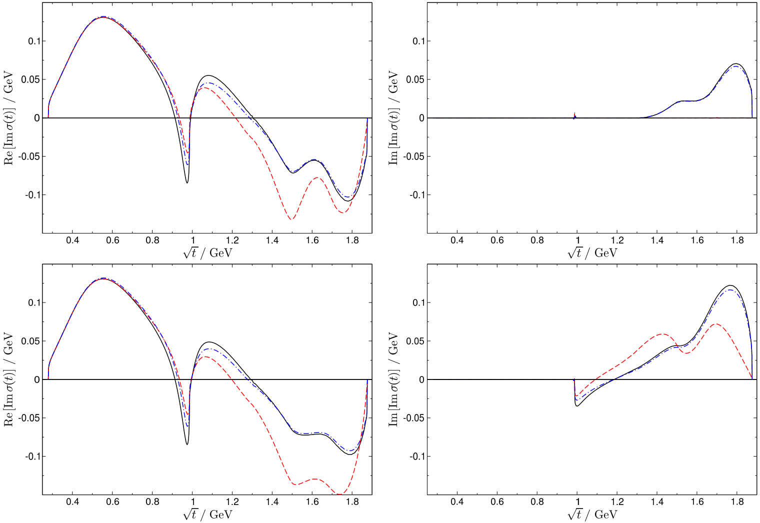
- Dispersion relation

$$\sigma(t) = \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} \mathrm{d}t'\,\frac{\mathrm{Im}\,\sigma(t')}{t'-t} = \sigma_{\pi N} + \frac{t}{\pi}\int\limits_{4M_\pi^2}^{\infty} \mathrm{d}t'\,\frac{\mathrm{Im}\,\sigma(t')}{t'(t'-t)}$$

  - **Unsubtracted**: $\sigma_{\pi N} = \sigma(0)$
  - **Once-subtracted**: $\Delta_\sigma = \sigma(2M_\pi^2) - \sigma_{\pi N}$

Spectral function

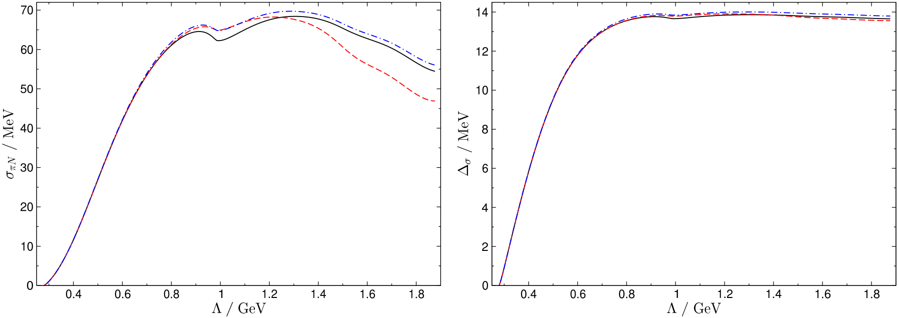


full

$h^0_+(t) = 0$

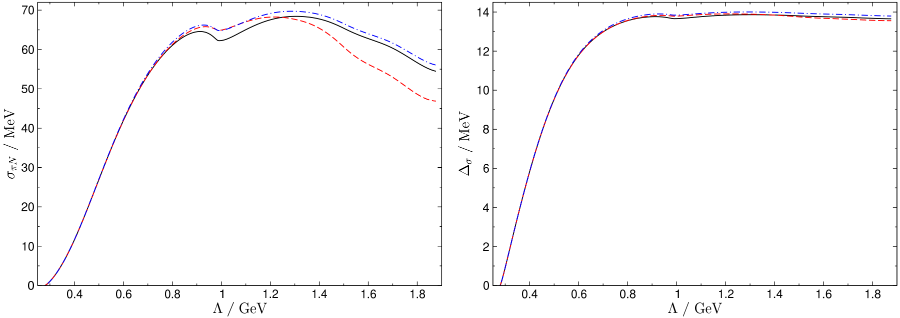
$\hookrightarrow$ Need $\bar{K}K$ states to ensure a **real imaginary part**

Convergence of the dispersive integral



- **Unsubtracted**: slow convergence
- **Once-subtracted**: stable result for $\Lambda \gtrsim 1\,\text{GeV}$

Convergence of the dispersive integral



- **Unsubtracted**: slow convergence
- **Once-subtracted**: stable result for $\Lambda \gtrsim 1\,\text{GeV}$
- Result for Δ_σ depends on pion–nucleon parameters

$$\Delta_\sigma = (13.9 \pm 0.3)\,\text{MeV}$$

$$+ Z_1\left(\frac{g^2}{4\pi} - 14.28\right) + Z_2\left(d_{00}^+ M_\pi + 1.46\right) + Z_3\left(d_{01}^+ M_\pi^3 - 1.14\right) + Z_4\left(b_{00}^+ M_\pi^3 + 3.54\right)$$

$$Z_1 = 0.36\,\text{MeV} \qquad Z_2 = 0.57\,\text{MeV} \qquad Z_3 = 12.0\,\text{MeV} \qquad Z_4 = -0.81\,\text{MeV}$$

Next correction: Δ_D

t-channel expansion

$$\bar{D}^+(\nu=0,t) = 4\pi\Big\{ -\frac{1}{p_t^2}\bar{f}_+^0(t) + \frac{5}{2}q_t^2\bar{f}_+^2(t) - \frac{27}{8}p_t^2q_t^4\bar{f}_+^4(t) + \frac{65}{16}p_t^4q_t^6\bar{f}_+^6(t) + \cdots \Big\}$$

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- Insert t-channel RS equations for Born-term-subtracted amplitudes $\bar{f}_+^J(t)$

$$\bar{D}^+(\nu=0,t)=d_{00}^+ + d_{01}^+ t - 16t^2\int\limits_{4M_\pi^2}^{\infty}\mathrm{d}t'\,\frac{\mathrm{Im}\,f_+^0(t')}{t'^2(t'-4m^2)(t'-t)}+\{J\geq 2\}+\{s\text{-channel integrals}\}$$

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t-channel expansion

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- $\Delta_D=F_\pi^2\Big(\bar{D}^+(\nu=0,t=2M_\pi^2)-d_{00}^+-2M_\pi^2d_{01}^+\Big)$ from evaluation at $t=2M_\pi^2$

$$\Delta_D=(12.1\pm0.3)\,\text{MeV}$$
$$+\tilde{Z}_1\bigg(\frac{g^2}{4\pi}-14.28\bigg)+\tilde{Z}_2\Big(d_{00}^+M_\pi+1.46\Big)+\tilde{Z}_3\Big(d_{01}^+M_\pi^3-1.14\Big)+\tilde{Z}_4\Big(b_{00}^+M_\pi^3+3.54\Big)$$

$$\tilde{Z}_1=0.42\,\text{MeV}\qquad \tilde{Z}_2=0.67\,\text{MeV}\qquad \tilde{Z}_3=12.0\,\text{MeV}\qquad \tilde{Z}_4=-0.77\,\text{MeV}$$

Summary: σ-term corrections

- Scalar form factor

$$\Delta_\sigma = (13.9 \pm 0.3)\,\text{MeV}$$
$$+Z_1\left(\frac{g^2}{4\pi} - 14.28\right) + Z_2\left(d_{00}^+ M_\pi + 1.46\right) + Z_3\left(d_{01}^+ M_\pi^3 - 1.14\right) + Z_4\left(b_{00}^+ M_\pi^3 + 3.54\right)$$
$$Z_1 = 0.36\,\text{MeV} \qquad Z_2 = 0.57\,\text{MeV} \qquad Z_3 = 12.0\,\text{MeV} \qquad Z_4 = -0.81\,\text{MeV}$$

- πN amplitude

$$\Delta_D = (12.1 \pm 0.3)\,\text{MeV}$$
$$+\tilde{Z}_1\left(\frac{g^2}{4\pi} - 14.28\right) + \tilde{Z}_2\left(d_{00}^+ M_\pi + 1.46\right) + \tilde{Z}_3\left(d_{01}^+ M_\pi^3 - 1.14\right) + \tilde{Z}_4\left(b_{00}^+ M_\pi^3 + 3.54\right)$$
$$\tilde{Z}_1 = 0.42\,\text{MeV} \qquad \tilde{Z}_2 = 0.67\,\text{MeV} \qquad \tilde{Z}_3 = 12.0\,\text{MeV} \qquad \tilde{Z}_4 = -0.77\,\text{MeV}$$

$\hookrightarrow$ most of the dependence on the πN parameters cancels in the difference!

Full correction

$$\Delta_D - \Delta_\sigma - \Delta_R = (-1.8 \pm 2.0)\,\text{MeV}$$

Origin of the cancellation

- Dominant contribution from dispersive integral over $f_+^0(t)$

$$\Delta_\sigma = \frac{3M_\pi^2}{\pi} \int\limits_{4M_\pi^2}^{\infty} dt' \frac{\sigma_{t'}^\pi \left(F_\pi^S(t')\right)^* f_+^0(t')}{t'(t'-2M_\pi^2)(4m^2-t')} + \cdots$$

$$\Delta_D = 64F_\pi^2 M_\pi^4 \int\limits_{4M_\pi^2}^{\infty} dt' \frac{\operatorname{Im} f_+^0(t')}{t'^2(t'-2M_\pi^2)(4m^2-t')} + \cdots = 64F_\pi^2 M_\pi^4 \int\limits_{4M_\pi^2}^{\infty} dt' \frac{\sigma_{t'}^\pi \left(t_0^0(t')\right)^* f_+^0(t')}{t'^2(t'-2M_\pi^2)(4m^2-t')} + \cdots$$

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- Largest contribution around $t' = 4M_\pi^2$

$$\frac{\Delta_\sigma}{\Delta_D} \rightarrow \frac{3M_\pi^2}{\pi} \frac{\left(F_\pi^S(t')\right)^* t'}{64F_\pi^2 M_\pi^4 \left(t_0^0(t')\right)^*} \rightarrow \frac{3M_\pi^4}{\pi} \frac{32\pi F_\pi^2 t'}{64F_\pi^2 M_\pi^4 (2t'-M_\pi^2)} \rightarrow \frac{6}{7} = \frac{18}{21} \qquad \text{ChPT: } \frac{\Delta_\sigma}{\Delta_D} = \frac{18}{23} + \mathcal{O}(M_\pi)$$

Origin of the cancellation

- Dominant contribution from dispersive integral over $f_+^0(t)$

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- Largest contribution around $t' = 4M_\pi^2$

$$\frac{\Delta_\sigma}{\Delta_D} \to \frac{3M_\pi^2}{\pi} \frac{\left(F_\pi^S(t')\right)^* t'}{64F_\pi^2 M_\pi^4 \left(t_0^0(t')\right)^*} \to \frac{3M_\pi^4}{\pi} \frac{32\pi F_\pi^2 t'}{64F_\pi^2 M_\pi^4 (2t'-M_\pi^2)} \to \frac{6}{7} = \frac{18}{21} \qquad \text{ChPT: } \frac{\Delta_\sigma}{\Delta_D} = \frac{18}{23} + \mathcal{O}(M_\pi)$$

- This explains
  - Δ_σ and Δ_D of similar size
  - Strong curvature generated by $\pi\pi$ rescattering

    $\hookrightarrow$ sensitivity to $\pi\pi$ phase shift reduced in the difference Gasser, Leutwyler, Sainio 1991
  - Spectral functions depend similarly on $f_+^0(t)$ $\hookrightarrow$ sensitivity to πN parameters reduced

Conclusions

- **Two-channel Muskhelishvili–Omnès problem with finite matching point**

  $\hookrightarrow \bar{N}N \to \pi\pi$ and $\bar{N}N \to \bar{K}K$ S-waves
- Dispersive analysis of the scalar form factor of the nucleon including $\bar{K}K$ effects in
  - Unitarity relation
  - Input for meson–baryon partial waves
  - Input for meson form factors
- Update $\Delta_D - \Delta_\sigma$ using modern phase shifts, **full two-channel treatment**

  $\hookrightarrow$ shift by 1.5 MeV compared to Gasser, Leutwyler, Sainio 1991
- Sensitivity to πN parameters negligible $\hookrightarrow$ cancels between Δ_D and Δ_σ