Analyticity and unitarity constraints on form factors

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Based on papers with G. Abbas, B. Ananthanarayan, C. Bourrely,
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Outline

1. Analyticity and unitarity for form factors
2. Method of "unitarity bounds" Okubo (1971)
3. Meiman problem and generalizations
4. Applications
5. Conclusions

Form factors considered

Electromagnetic form factors of light pseudoscalar mesons: $P = \pi, K$

- $\langle P^+(p')|J_\mu^{\rm elm}|P^+(p)\rangle = (p+p')_\mu F_P(t)$

Form factors relevant for weak semileptonic transitions: $P \to \pi$, $P = K, D, B$

- $\langle \pi^+(p')|J_\mu^{\rm weak}|P^0(p)\rangle = (p'+p)_\mu f_+(t) + (p-p')_\mu f_-(t)$
    - $f_+(t)$: vector form factor
    - $f_0(t) = f_+(t) + \frac{t}{M_P^2 - M_\pi^2} f_-(t)$: scalar form factor

In the general discussion $F_P(t)$ and $f_\pm(t)$ shall be denoted generically as $F(t)$

Basic properties

Theoretical description of $F(t)$

- at low energies: ChPT, lattice, QCD-SR
- at high $t = -Q^2 < 0$ perturbative QCD ($1/t$ scaling)
- intermediate region: big uncertainties

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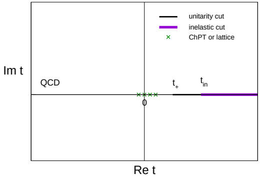

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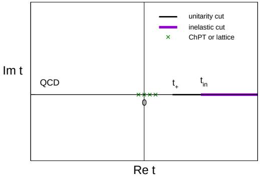

Analyticity and unitarity

- Causality: $F(t)$ real analytic function, $F(t^*) = F^*(t)$, in the complex t-plane with a cut along the real axis from the lowest unitarity threshold t_+ to infinity
- Unitarity: $\mathrm{Im}F(t+i\epsilon) = \theta(t-t_+)\,\sigma(t)f^*(t)F(t) + \theta(t-t_{in})\,\Sigma_{in}(t)$

$$\sigma(t) = \sqrt{1 - t_+/t}\text{: two particle phase space}$$

$\Rightarrow$ Fermi-Watson theorem: for $t_+ \leq t \leq t_{in}$, $\arg[F(t+i\epsilon)] = \delta(t)$,

$\delta(t)$: phase-shift of the related scattering amplitude $f(t) = \frac{e^{2i\delta(t)}-1}{2i\sigma(t)}$

- Complications: unphysical regions, anomalous thresholds (not encountered)

- Dispersive representations

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  - Standard dispersion relation (Cauchy integral)

$$F(t) = \frac{1}{\pi} \int_{t_+}^{\infty} \frac{\mathrm{Im} F(t'+i\epsilon) dt'}{t'-t} \qquad \text{(modulo subtractions)}$$

$$\mathrm{Im}\, F(t+i\epsilon) = \sigma(t) f^*(t) F(t) \quad \text{for } t < t_{in}$$

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  - Omnès (phase) representations

$$F(t) = P(t) \exp\left(\frac{t}{\pi} \int_{t_+}^{\infty} dt' \frac{\delta(t')}{t'(t'-t)}\right)$$

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  - Representation in terms of modulus

$$F(t) = B(t)\exp\left(\frac{\sqrt{t_+-t}}{\pi}\int_{t_+}^{\infty}\frac{\ln|F(t')|\,\mathrm{d}t'}{\sqrt{t'-t_+}(t'-t}\right)$$

$B(t)$: Blaschke factor ($|B(t)| = 1$ for $t > t_+$, $B(t_i) = 0$)

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- Analytic parametrizations - little predictive power outside their original range

Okubo's approach

"Method of unitarity bounds" Okubo (1971), Micu (1973), Auberson et al (1975), Singh and Raina (1979)

- Polarization tensor of the relevant current calculated from current algebra at spacelike momenta
- Dispersion relation for the invariant polarization amplitudes
- Unitarity and positivity of the spectral functions

  $\Rightarrow$ an upper bound on an integral of the modulus squared of the form factor along the unitarity cut

  $\Rightarrow$ mathematical techniques of complex analysis lead to bounds on the values of the form factor and its derivatives at points inside the analyticity domain

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- Modern version: the spacelike correlators are obtained from perturbative QCD and OPE Bourrely, Machet, de Rafael (1981), de Rafael and Taron (1992)

- Polarization tensor of a relevant weak current V_μ:

$$i\int d^4x\, e^{iq\cdot x}\langle 0|\, T\left\{V^\mu(x)V^\nu(0)^\dagger\right\}|0\rangle = (-g^{\mu\nu}q^2+q^\mu q^\nu)\Pi_1(q^2)+q^\mu q^\nu\Pi_0(q^2)$$

- Unsubtracted dispersion relations for suitable correlators

$$\chi_1(Q^2)\equiv -\frac{1}{2}\frac{\partial^2}{\partial(Q^2)^2}\left[Q^2\Pi_1(-Q^2)\right]=\frac{1}{\pi}\int_0^\infty dt\,\frac{t\mathrm{Im}\Pi_1(t)}{(t+Q^2)^3}$$

$$\chi_0(Q^2)\equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_0(-Q^2)\right]=\frac{1}{\pi}\int_0^\infty dt\,\frac{t\mathrm{Im}\Pi_0(t)}{(t+Q^2)^2}$$

- Unitarity and the positivity of the spectral functions ($t_\pm=(M_P\pm M_\pi)^2$)

$$\mathrm{Im}\Pi_1(t)\geq \tfrac{3}{2}\tfrac{1}{48\pi}\frac{[(t-t_+)(t-t_-)]^{3/2}}{t^3}|f_+(t)|^2$$

$$\mathrm{Im}\Pi_0(t)\geq \tfrac{3}{2}\tfrac{t_+t_-}{16\pi}\tfrac{[(t-t_+)(t-t_-)]^{1/2}}{t^3}|f_0(t)|^2$$

$\Rightarrow$ $\mathbf{\frac{1}{\pi}\int_{t_+}^{\infty}\rho(t,Q^2)|F(t)|^2 dt\leq I(Q^2)}$, $I(Q^2)$ calculated from pQCD and OPE

Illustration

- Polarization tensor of a relevant weak current V_μ:

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- Q^2 sufficiently large for light mesons; $Q^2=0$ for heavy-heavy or heavy-light form factors
- more general relation: $\frac{1}{\pi}\int_{t_+}^\infty \rho_{i,j}(t,Q^2)F_i(t)F_j^*(t)dt \leq I(Q^2)$ ($BD^{(*)}$ form factors)

Comment: connection with stability of analytic continuation

Analyticity: its splendour and its dangers

Splendour: analytic continuation is unique

Dangers: analytic continuation is unstable (ill posed problem in the Hadamard sense)

- two analytic functions very close along a range Γ may differ arbitrarily outside Γ
  - determination of remote resonances from Breit-Wigner parametrizations!

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Mathematical result (Tikhonov regularization): analytic continuation is stabilized if the class of admissible functions forms a compact set   Ciulli et al (1975)

- Role of the stabilizing condition:
  - Let $\mathcal{C}$ be a compact class of analytic functions
  - If $F_j(t) \in \mathcal{C}$ and $\sup_\Gamma |F_1(t) - F_2(t)| < \epsilon$, then for t outside Γ the inequality $|F_1(t) - F_2(t)| < M(\epsilon, t)$ holds, such that $M(\epsilon, t) \to 0$ when $\epsilon \to 0$

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Important remark:

- The inequality derived from Okubo's approach defines a compact set in the Hardy space H^2 of analytic functions with finite L^2 norm on the boundary

  $\Rightarrow$ this ensures the stability of extrapolation to points inside the holomorphy domain

Consequences of the integral condition

Problem 1: From the L^2-norm condition

$$\frac{1}{\pi}\int_{t_+}^{\infty} \rho(t)|F(t)|^2 dt \leq I$$

find constraints on the values of the values $F(t_n)$ and the derivatives $F^{(k)}(t_j)$ at some real or complex points

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Write the problem in a canonical form:

- Conformal mapping of the t-plane cut for $t > t_+$ onto a unit disk by $z \equiv \tilde{z}(t, t_0)$, with the inverse $\tilde{t}(z, t_0)$

$$\tilde{z}(t, t_0) = \frac{\sqrt{t_+ - t_0} - \sqrt{t_+ - t}}{\sqrt{t_+ - t_0} + \sqrt{t_+ - t}}, \qquad \tilde{z}(t_0, t_0) = 0$$

- Define an outer function $w(z)$, *i.e.* analytic and without zeros in $|z| < 1$, with modulus squared on $|z| = 1$ equal to $\rho(t)|d\tilde{t}/dz|$:

$$w(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \, \frac{e^{i\theta}+z}{e^{i\theta}-z} \, \ln[\rho(\tilde{t}(e^{i\theta}))|d\tilde{t}/dz|]\right]$$

$\Rightarrow$ the function $g(z) = F(\tilde{t}(z, t_0))\, w(z)$ is analytic in $|z| < 1$ and satisfies the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2 d\theta \leq I$$

If $g(z)$ analytic in $|z| < 1$ and $\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2 d\theta \leq I$

$$\left[\frac{1}{k!}\frac{d^k g(z)}{dz^k}\right]_{z=0} = g_k, \quad 0 \leq k \leq K-1, \qquad g(z_n) = \xi_n, \quad z_n = z_n^*, \quad 1 \leq n \leq N$$

$$\bar{I} = I - \sum_{k=0}^{K-1} g_k^2, \qquad \bar{\xi}_n = \xi_n - \sum_{k=0}^{K-1} g_k z_n^k$$

$\Rightarrow$ positivity of the following determinant and of its minors:

$$\begin{vmatrix} \bar{I} & \bar{\xi}_1 & \bar{\xi}_2 & \cdots & \bar{\xi}_N \\ \bar{\xi}_1 & \frac{z_1^{2K}}{1-z_1^2} & \frac{(z_1 z_2)^K}{1-z_1 z_2} & \cdots & \frac{(z_1 z_N)^K}{1-z_1 z_N} \\ \bar{\xi}_2 & \frac{(z_1 z_2)^K}{1-z_1 z_2} & \frac{(z_2)^{2K}}{1-z_2^2} & \cdots & \frac{(z_2 z_N)^K}{1-z_2 z_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\xi}_N & \frac{(z_1 z_N)^K}{1-z_1 z_N} & \frac{(z_2 z_N)^K}{1-z_2 z_N} & \cdots & \frac{z_N^{2K}}{1-z_N^2} \end{vmatrix} \geq 0$$

Alternative solution based on analytic interpolation theory

Convex domain in the space of parameters, defined by the quadratic inequality:

$$\sum_{m,n=1}^{N} \mathcal{A}_{mn}\xi_n\xi_m + \sum_{j,k=0}^{K-1} \mathcal{B}_{jk} g_j g_k + 2\sum_{n=1}^{N}\sum_{k=0}^{K-1} \mathcal{C}_{kn} g_k \xi_n \leq I$$

- Blaschke factors: $|B_j(z)| = 1$ for $|z| = 1$ defined recurrently by

$$B_1(z) = 1, \quad B_n(z) = \frac{z - z_{n-1}}{1 - z z_{n-1}}\, B_{n-1}(z), \quad 2 \leq n \leq N+1,$$

- $\beta_{kl} = \frac{1}{(K+l-k)!} \frac{d^{K+l-k}}{dz^{K+l-k}} \left[\frac{1}{B_{N+1}(z)}\right]_{z=0}, \qquad Y_n = \left[\frac{z - z_n}{B_{N+1}(z)}\right]_{z=z_n}$

$$\mathcal{A}_{mn} = \frac{Y_n Y_m}{z_n^K z_m^K} \frac{1}{1 - z_n z_m}, \qquad \mathcal{B}_{jk} = \sum_{l=L}^{-1} \beta_{jl}\beta_{kl}$$

$$\mathcal{C}_{kn} = \frac{Y_n}{z_n^K} \sum_{l=k-K}^{-1} \frac{\beta_{kl}}{z_n^{l+1}}$$

Inclusion of the phase in the elastic region

Problem 2: From the conditions

$$\frac{1}{\pi}\int_{t_+}^{\infty}\rho(t)|F(t)|^2 dt \leq I \quad \text{and} \quad \arg[F(t+i\epsilon) = \delta(t), \quad t_+ \leq t \leq t_{in}$$

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Standard techniques of functional optimization $\Rightarrow$ solution described by the inequality:

$$\sum_{m,n=1}^{N}\mathcal{A}_{mn}\xi_n\xi_m + \sum_{j,k=0}^{K-1}\mathcal{B}_{jk}g_j g_k + 2\sum_{n=1}^{N}\sum_{k=0}^{K-1}\mathcal{C}_{kn}g_k\xi_n + \frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} {\rm d}\theta\lambda(\theta)V(\theta) \leq I$$

$\lambda(\theta)$: the solution of a Fredholm integral equation

- $\lambda(\theta) - \frac{1}{2\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} {\rm d}\theta'\lambda(\theta')\mathcal{K}_\Psi(\theta,\theta') = V(\theta), \qquad e^{i\theta_{in}} = \tilde{z}(t_{in}, t_0)$
- $\mathcal{K}_\Psi(\theta,\theta') \equiv \frac{\sin[(K-1/2)(\theta-\theta')-\Psi(\theta)+\Psi(\theta')]}{\sin[(\theta-\theta')/2]}$
- $\Psi(\theta)$ and $V(\theta)$: known functions depending linearly on the input values

Modified analytic optimization problem

Problem 3: From the conditions

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Steps of the proof:

- Define the Omnès function (for $t > t_{in}$, $\delta(t)$ arbitrary smooth function):

$$\mathcal{O}(t) = \exp\left(\frac{t}{\pi}\int_{t_+}^{\infty} dt \frac{\delta(t')}{t'(t'-t)}\right)$$

- Define the function $h(t)$ by $F(t) = \mathcal{O}(t)h(t)$, with the properties:
  - $h(t)$ is real below t_{in}, *i.e.* is analytic in the t-plane cut only for $t > t_{in}$
  - $\frac{1}{\pi}\int_{t_{in}}^{\infty} \rho(t)|\mathcal{O}(t)|^2|h(t)|^2 dt \leq I'$

$\Rightarrow$ for $h(t)$ we obtained Problem 1, with two modifications:

- the t-plane cut for $t > t_+$ is replaced by the t-plane cut for $t > t_{in}$
- the weight $\rho(t)$ is replaced by $\rho(t)|\mathcal{O}(t)|^2$

Solution of Problem 3

- Conformal mapping of the t-plane cut for $t > t_{in}$ onto the unit disc $|z| < 1$:

$$\tilde{z}(t, t_0) = \frac{\sqrt{t_{in} - t_0} - \sqrt{t_{in} - t}}{\sqrt{t_{in} - t_0} + \sqrt{t_{in} - t}}, \qquad \tilde{z}(t_0, t_0) = 0$$

- Outer functions with modulus related to $\rho(t)$ and $|\mathcal{O}(t)|$:

$$w(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \, \frac{e^{i\theta} + z}{e^{i\theta} - z} \, \ln[\rho(\tilde{t}(e^{i\theta}, t_0))|d\tilde{t}/dz|]\right]$$

$$\omega(z) = \exp\left(\frac{\sqrt{t_{in} - \tilde{t}(z, t_0)}}{\pi} \int_{t_{in}}^{\infty} \mathrm{d}t' \, \frac{\ln|\mathcal{O}(t')|}{\sqrt{t' - t_{in}}(t' - \tilde{t}(z, t_0))}\right)$$

$\Rightarrow$ the function $g(z)$ defined by:

$$g(z) \equiv F(\tilde{t}(z, t_0)) \, [\mathcal{O}(\tilde{t}(z, t_0))]^{-1} \, w(z) \, \omega(z)$$

is analytic in $|z| < 1$ and satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^2 d\theta \leq I'$$

$\Rightarrow$ the standard Meiman problem

Properties of the bounds

Rigorous properties:

- are independent of the conformal mapping (the parameter t_0) and the arbitrary phase $\delta(t)$ for $t > t_{in}$
- remain the same if the $\leq$ sign is replaced by the equality sign
- depend in a monotonous way on I: larger I, weaker constraints

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Related problems: Nevanlinna-Pick and Schur-Carathéodory interpolation for bounded functions

$$||F||_{L^\infty} \equiv \sup_{t>t_+} |F(t)| \leq I$$

- The bounds in L^∞-norm are stronger than those based in L^2-norm
- By varying $\rho(t)$, we can approach the stronger bounds given by the L^∞-norm

  $\Rightarrow$ hints for a suitable choice of $\rho(t)$: compromise between
  - choices leading to strong bounds
  - need to exploit properly the available knowledge of the modulus

Particular consequence: domains without zeros

The formalism predicts domains in the t-plane where zeros are excluded

- Insert the assumption $F(t_c)=0$ in the determinant along with other input values
- If the consistency inequality is violated, the zero is excluded

  $\Rightarrow$ rigorous description of the domains where zeros are forbidden

- Example: given the input values $F(0)$, $F'(0)$ and $F(t_1)$, the domain of points $t_c = \tilde{t}(z_c, t_0)$ where zeros of $F(t)$ are excluded is defined by the inequality

$$\begin{vmatrix} I - g_0^2 - g_1^2 & -g_0 - g_1 z_c & g(z_1) - g_0 - g_1 z_1 \\ -g_0 - g_1 z_c & \frac{z_c^4}{1-z_c^2} & \frac{(z_c z_1)^2}{1-z_c z_1} \\ g(z_1) - g_0 - g_1 z_1 & \frac{(z_0 z_1)^2}{1-z_0 z_1} & \frac{(z_1)^4}{1-z_1^2} \end{vmatrix} < 0$$

The knowledge of zeros is important for testing symmetry properties and as input in some dispersive representations

Applications

1. Constraints on the low energy parameters and zeros
   - $BD^{(*)}$ form factors (Isgur-Wise function) IC, Lellouch, Neubert (1998)
   - $K\pi$ form factors Bourrely, IC (2005), Abbas, Ananthanarayan, IC, Imsong, Ramanan (2010) (Anant's talk)
   - pion electromagnetic form factor IC (2000), Abbas, Anant, IC, Imsong (2011)
   - $D\pi$ form factors Ananthanarayan, IC, Imsong (2011)

2. Extrapolations to intermediate spacelike energies
   - onset of pQCD for the pion form factor Ananthanarayan, IC, Imsong (2012)

3. Bounds on the modulus on the timelike axis
   - consistency checks on pion form factor data Ananthanarayan, IC, Das, Imsong (work in progress)

4. Analytic parametrizations with unitarity constraints
   - $BD^{(*)}$ form factors (Isgur-Wise function) IC, Lellouch, Neubert (1998)
   - $B\pi$ vector form factor Bourrely, IC, Lellouch (2009)
   - $D\pi$ form factors, pion electromagnetic form factor (work in progress)

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   - $K\pi$ form factors Bourrely, IC (2005), Abbas, Ananthanarayan, IC, Imsong, Ramanan (2010) (Anant's talk)
   - pion electromagnetic form factor IC (2000), Abbas, Anant, IC, Imsong (2011)
   - $D\pi$ form factors Ananthanarayan, IC, Imsong (2011)

2. Extrapolations to intermediate spacelike energies
   - onset of pQCD for the pion form factor Ananthanarayan, IC, Imsong (2012)

3. Bounds on the modulus on the timelike axis
   - consistency checks on pion form factor data Ananthanarayan, IC, Das, Imsong (work in progress)

4. Analytic parametrizations with unitarity constraints
   - $BD^{(*)}$ form factors (Isgur-Wise function) IC, Lellouch, Neubert (1998)
   - $B\pi$ vector form factor Bourrely, IC, Lellouch (2009)
   - $D\pi$ form factors, pion electromagnetic form factor (work in progress)

Historical review and references in: G. Abbas, B. Ananthanarayan, IC, I.S. Imsong and S. Ramanan, Eur. Phys. J. A **45**, 389 (2010), arXiv:1004.4257 [hep-ph]

Ananthanarayan, IC, Imsong, EPJ A 47, 147 (2011)

Of interest for the determination of the element $|V_{cd}|$ of the CKM matrix

Input:

- $f_+(0) = 0.67 \pm 0.10$, from LCSR Khodjamirian et al (2009) and lattice HPQCD (2011)
- low-energy soft-pion theorem (Callan-Treiman): $f_0(M_D^2 - M_\pi^2) = f_D/f_\pi$ Dominguez et al (1990)
- phase at low energies from dominant resonances D^* and D_0^*
- Okubo's approach: derivatives $\chi_k^{(n)}$ of a polarization function at $Q^2 = 0$

$$\frac{1}{\pi}\int_{t_+}^{\infty} \rho_k^{(n)}(t)|f_k(t)|^2 dt \leq \chi_k^{(n)}, \qquad k = +, 0$$

$$\chi_k^{(n)} = \chi_k^{(n)PT} + \chi_k^{(n)NP}:$$ pQCD to two loops Chetyrkin et al (2001)

n	$\chi_+^{(n)PT}$	$\chi_+^{(n)NP}$	$\chi_+^{(n)}$	$\chi_0^{(n)PT}$	$\chi_0^{(n)NP}$	$\chi_0^{(n)}$
0	0.0170744	-0.0010543	0.0160201	0.0045547	0.0002723	0.0048270
1	0.0019357	-0.0002723	0.0016634	0.0004118	0.0000704	0.0004821
2	0.0002586	-0.0000704	0.0001883	0.0000524	0.0000182	0.0000706

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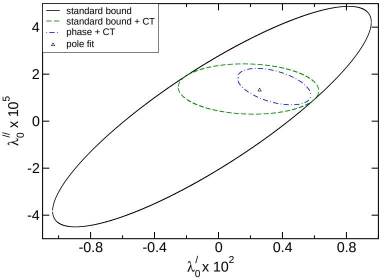
Taylor expansion at $t = 0$:

$$f_k(t) = f_k(0)\left(1 + \lambda_k' \frac{t}{M_\pi^2} + \frac{1}{2}\lambda_k'' \frac{t^2}{M_\pi^4} + \cdots\right), \qquad k = +, 0$$

Scalar form factor, moment $\chi_0^{(0)}$



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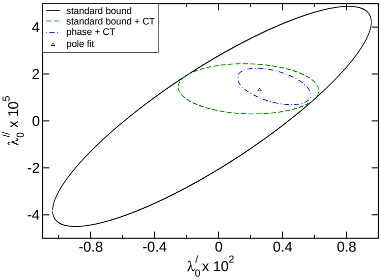
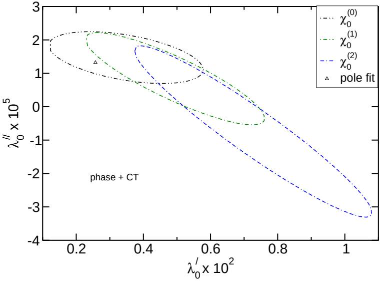


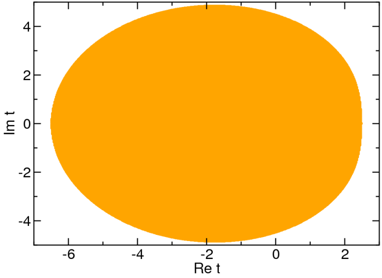
Constraints from various moments



Intersection $\Rightarrow$ small allowed domain

- Pole ansatz from Becirevic, Kaidalov (1999) excluded by imposing all the constraints

Vector form factor, moment $\chi_1^{(0)}$



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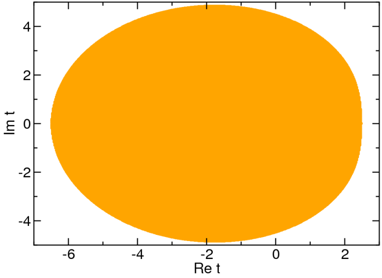
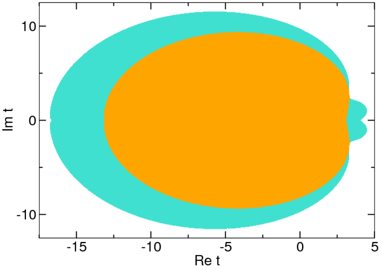


Scalar form factor, moment $\chi_0^{(0)}$

$\Rightarrow$ larger region using CT theorem



Application I: Low-energy constraints on the pion form factor

Ananthanarayan, IC, Imsong, Phys Rev D83, 096002 (2011)

Input:

- $\delta(t) = \delta_1^1(t)$ for $t \leq t_{in} = (M_\pi + M_\omega)^2 = (0.917 \text{ GeV})^2$ from Roy equations for $\pi\pi$ amplitude Ananthanarayan et al (2001), Garcia-Martin et al (2011)
- Recent measurements of the modulus up to high energies BaBar (2009)
- Precise measurements at spacelike points Horn et al (2008), Huber et al (2008)

t	Value [GeV^2]	$F(t)$
t_1	-1.60	$0.243 \pm 0.012^{+0.019}_{-0.008}$
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Best results obtained from Problem 3: $\frac{1}{\pi}\int_{t_{in}}^{\infty} \rho(t)|F(t)|^2 dt \leq I'$

Example: suitable choice of $\rho(t)$:

- $\rho(t) = \rho_\mu(t) = \frac{\alpha^2 M_\mu^2}{12\pi} \frac{(t-t_+)^{3/2}}{t^{7/2}} K(t), \qquad K(t) = \int_0^1 du\, \frac{(1-u)u^2}{1-u+M_\mu^2 u^2/t}$
- $I' \equiv \hat{a}_\mu^{\pi\pi} = 22.17 \times 10^{-10}$ Davier et al (2010), Malaescu (private communication)

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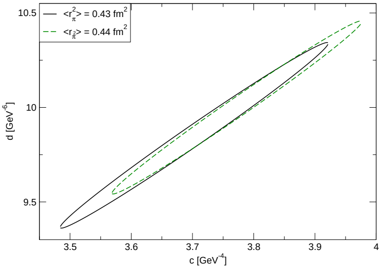
Taylor expansion: $F(t) = 1 + \frac{1}{6}\langle r_\pi^2\rangle t + c t^2 + d t^3 + \cdots$

$$\langle r_\pi^2\rangle = 0.43 \pm 0.01\ \text{fm}^2$$

Low-energy constraints: allowed domain in the c-d plane



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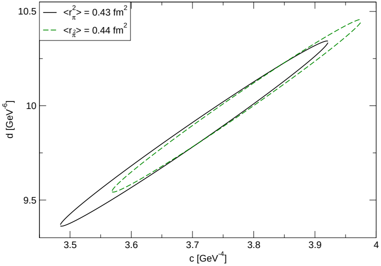

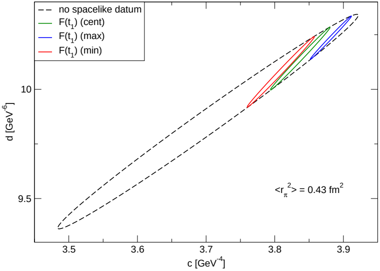

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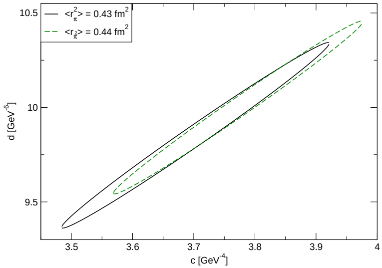

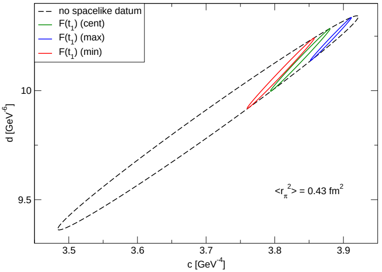

By varying all the input parameters, $\langle r_\pi^2 \rangle, F(t_1), \hat{a}_\mu^{\pi\pi}, \delta_1^1(t)$:

$$3.75 \text{ GeV}^{-4} \lesssim c \lesssim 3.98 \text{ GeV}^{-4}, \qquad 9.91 \text{ GeV}^{-6} \lesssim d \lesssim 10.46 \text{ GeV}^{-6}$$

with a strong correlation between the values of c and d

Pion form factor: domains without zeros

$\langle r_\pi^2 \rangle = 0.43 \pm 0.01\,\mathrm{fm}^2$ and $F(t_1) = 0.234^{+0.022}_{-0.014}$ $\Rightarrow$ simple zeros on the real axis are excluded in the range $-4.46\ \mathrm{GeV}^2 \leq t_0 \leq 0.84\ \mathrm{GeV}^2$

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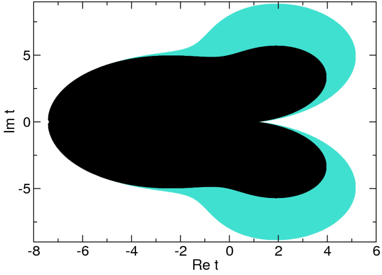

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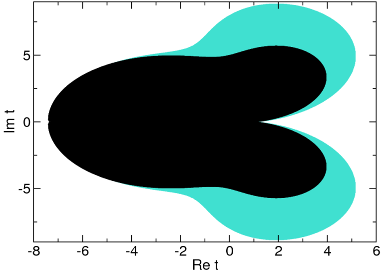
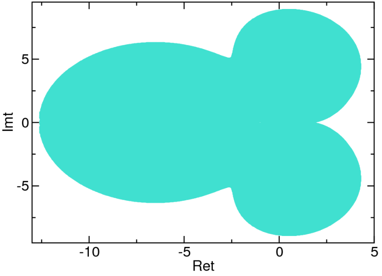


Complex zeros, with spacelike input:
$F(t_1) = 0.234$, $\langle r_\pi^2 \rangle = 0.43\,\mathrm{fm}^2$



Application II: Bounds on the pion form factor on the spacelike axis

Ananthanarayan, IC, Imsong, Phys Rev D85, 096006 (2012)

Input:

- $F(0) = 1, \quad \langle r_\pi^2 \rangle = 0.43 \pm 0.01\,\text{fm}^2, \quad F(-2.45\,\text{GeV}^2) = 0.167 \pm 0.010^{+0.013}_{-0.007}$
- $\arg[F(t+i\epsilon)] = \delta_1^1(t), \qquad 4M_\pi^2 \le t \le t_{in}, \quad t_{in} = (M_\omega + M_\pi)^2$
- $\frac{1}{\pi}\int\limits_{t_{in}}^{\infty} \rho(t)|F(t)|^2 dt \le I'$, for suitable choices of the weight $\rho(t)$

Direct evaluation of I':

- BaBar data Aubert et al (2009) up to 3 GeV
- very conservative estimate above 3 GeV, imposing the decrease $|F(t)| \sim 1/t$ above 20 GeV

  $\Rightarrow$ small sensitivity to the high-energy assumptions for weights that decrease as $1/\sqrt{t}$ or faster

Ananthanarayan, IC, Imsong, Phys Rev D85, 096006 (2012)

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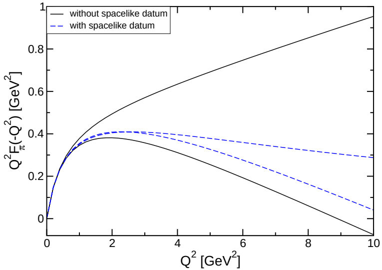
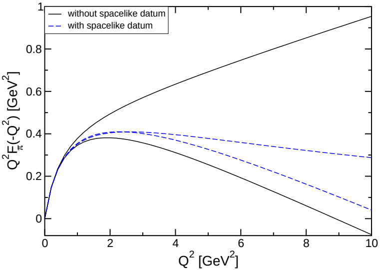
$\rho(t)$	I'
1	1.788 ± 0.039
$1/\sqrt{t}$	0.687 ± 0.028
$1/t$	0.578 ± 0.022
$1/t^2$	0.523 ± 0.017

$\rho(t) = 1/\sqrt{t}$, inclusion of $F(t_2)$

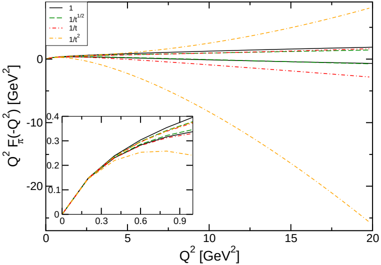


Choice of the best weight

$\rho(t) = 1/\sqrt{t}$, inclusion of $F(t_2)$



Comparison of various weights



- weights that decrease too fast have a weak constraining power at large energies
  - the weight related to the standard Okubo approach and ρ_μ, which decrease like $1/t^2$, are not useful for extrapolation to large Q^2
- weights that decrease too slowly are sensitive to the asymptotic tail

$\Rightarrow$ suitable choice: $\rho(t) = 1/\sqrt{t}$

Effect of the uncertainty of the input

Including the errors:

- vary separately each input and combine the resulting errors in quadrature
- vary simultaneously all the input quantities inside their error intervals

⇒ lead to comparable results

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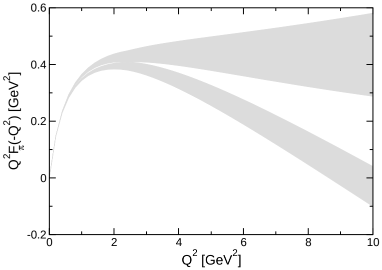
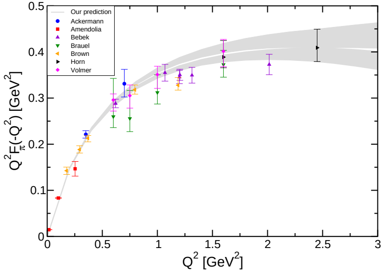


Bounds for the optimal choice $\rho = \frac{1}{\sqrt{t}}$

- white band: allowed domain for central values of the input variables
- grey bands: enlarged domain when the input is varied inside the error bars

Comparison with experimental data

Low energy data



- A few points (Amendolia, Bebek) in conflict with the bounds

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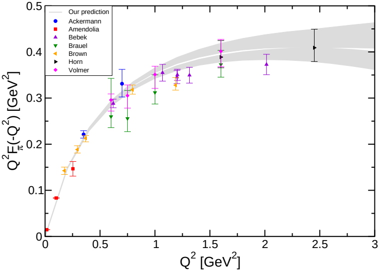
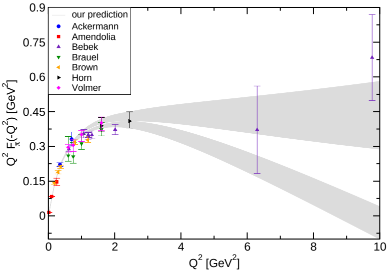


High energy data



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Comparison with pQCD and nonperturbative theoretical models

$$F_{\rm QCD}^{\rm LO}(-Q^2) = \frac{8\pi f_\pi^2 \alpha_s(\mu^2)}{Q^2}$$

$$F_{\rm QCD}^{\rm NLO}(-Q^2) = \frac{8 f_\pi^2 \alpha_s^2(\mu^2)}{Q^2} \left[\frac{\beta_0}{4}\left(\ln\frac{\mu^2}{Q^2} + \frac{14}{3}\right) - 3.92\right]$$

The asymptotic regime is known to set in quite slowly

- Various nonperturbative models for the intermediate regions
- Definite conclusions difficult due to lack of data at high energy

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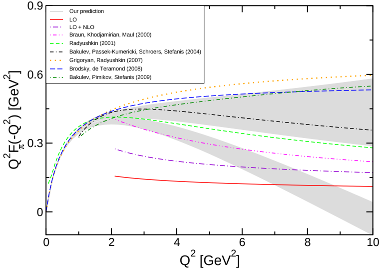

Ananthanarayan, IC, Das, Imsong, preliminary results

- The formalism applied up to now (Problem 3) does not use as input data on the modulus $|F(t)|$ for $t < t_{in}$
- On the other hand, the formalism allows to find bounds on this quantity

- The function

$$g(z) \equiv F(\tilde{t}(z, t_0))\,[\mathcal{O}(\tilde{t}(z, t_0))]^{-1}\, w(z)\,\omega(z)$$

is analytic and real for $t < t_{in}$

- Derive upper and lower bounds on $g(z)$ from Meiman condition
- They lead to bounds on the modulus $F(t)$, calculated as

$$|F(t)| = |\mathcal{O}(t)|\,\frac{g(\tilde{z}(t_0, t))}{w(\tilde{z}(t_0, t))}$$

$$|\mathcal{O}(t)| = \exp\left(\frac{t}{\pi} PV \int_{t_+}^{\infty} dt \frac{\delta(t')}{t'(t'-t)}\right)$$

Isospin breaking by ω-ρ interference in e^+e^- annihilation:

$$\delta_1^1(t) \Rightarrow \delta_1^1(t) + \arg\left[1 + \frac{\epsilon t}{t_\omega - t}\right], \quad t_\omega = (M_\omega - i/2\,\Gamma_\omega)^2, \quad \epsilon \approx 1.9 \times 10^{-3}$$

Leutwyler (2002), Hanhart (2012)

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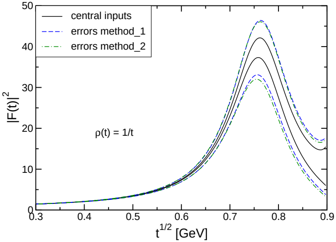

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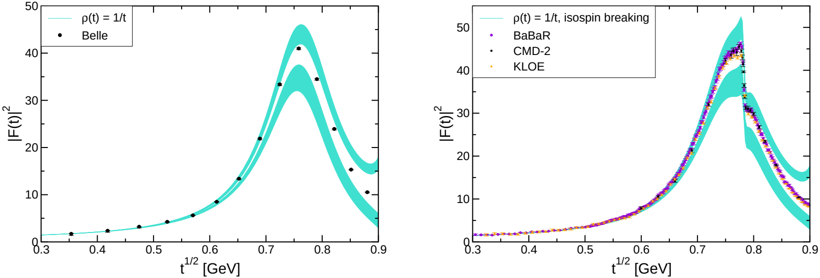

- accurate recent data: Belle (2008), BaBar (2009), KLOE (2011), CMD-2 (2007)

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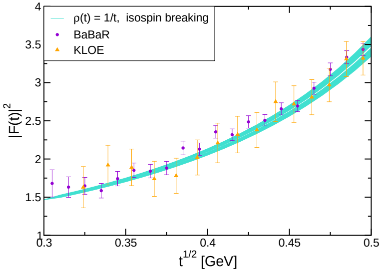


- The bounds describe more precisely the modulus at low energies than the data
- A few experimental points (BaBar) in conflict with the bounds

Application IV: Parametrization of the $B\pi$ vector form factor

Bourrely, IC, Lellouch, arXiv:0807.2772, Phys Rev D79, 013008 (2009)

Of interest for the determination of the element $|V_{ub}|$ of the CKM matrix

$$\frac{d\Gamma}{dq^2}(\bar{B}^0 \to \pi^+ l^- \bar{\nu}_\ell) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3 m_B^3} \lambda^{3/2}(q^2)|f_+(q^2)|^2$$

Physical range of semileptonic decays: $0 \leq q^2 \leq t_- \equiv (m_{B^0} - m_{\pi^+})^2 = 26.42\,\mathrm{GeV}^2$

Basic properties:

- $f_+(q^2)$ analytic in the q^2-plane cut for $q^2 \geq t_+$, with $t_+ = (m_{B^0} + m_{\pi^+})^2$ except for a pole at $q^2 = M_{B*}^2$
- Threshold behaviour: $\mathrm{Im}\, f_+(q^2) \sim (q^2 - t_+)^{3/2}$
- Unitarity constraint (Okubo): $\frac{1}{\pi}\int_{t_+}^{\infty} \rho(t)|f_+(t)|^2 dt \leq \chi_{1^-}(0)$

$$\chi_{1^-}(0) = \frac{3[1 + 1.14\,\alpha_s(\bar{m}_b)]}{32\pi^2 m_b^2} - \frac{\bar{m}_b\langle\bar{u}u\rangle}{m_b^6} - \frac{\langle\alpha_s G^2\rangle}{12\pi m_b^6} \approx 5.01 \times 10^{-4}$$

Generalis (1990), Lellouch (1996), Arnesen et al (2005)

$$f_+(q^2) = \frac{1}{1 - q^2/m_{B^*}^2} \sum_{k=0}^{K-1} b_k(t_0) \left[z^k - (-1)^{k-K} \frac{k}{K} z^K \right], \qquad z = \tilde{z}(q^2, t_0)$$

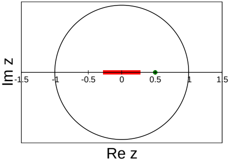
Unitarity constraint: $\displaystyle\sum_{j,k=0}^{K} B_{jk}(t_0) b_j(t_0) b_k(t_0) \leq 1$

$t_0(\mathrm{GeV}^2)$	B_{00}	B_{01}	B_{02}	B_{04}	B_{04}	B_{05}
0	0.0197	-0.0049	-0.0108	0.0057	0.0006	-0.0005
t_{opt}	0.0197	0.0042	-0.0109	-0.0059	-0.0002	0.0012
t_-	0.0197	0.0118	-0.0015	-0.0078	-0.0077	-0.0053

$$B_{j(j+k)} = B_{0k}, \qquad B_{jk} = B_{kj}$$

Optimal choice of t_0:

$$t_{opt} \equiv (m_B + m_\pi)(\sqrt{m_B} - \sqrt{m_\pi})^2 = 20.062\,\mathrm{GeV}^2$$



Fit of theoretical and experimental data

Theoretical and experimental input:

- $f_+(0) = 0.26 \pm 0.03$, from LCSR Khodjamirian et al (1007, Ball (2007)
- lattice calculations at eight q^2-points FNAL-MILC and HPQCD
- experimental data on the partial branching fractions over bins in q^2 BaBar, Belle, CLEO

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Combined fit of experimental and theoretical points, with unitarity constraint:

$$\mathcal{L}(b_j, |V_{ub}|) = \chi^2(b_j, |V_{ub}|) + \lambda \left(\sum_{j,k=0}^{K} B_{jk} b_j b_k - 1 \right)$$

$$\chi^2(b_k, |V_{ub}|) = \chi^2_{th} + \chi^2_{exp}$$

$$\chi^2_{th} = \sum_{j,k=1}^{8} [f_j^{in} - f_+(q_j^2)] C_{jk}^{-1} [f_k^{in} - f_+(q_k^2)] + (f_+(0) - f_{\rm LCSR})^2 / (\delta f_{\rm LCSR})^2$$

$$\chi^2_{exp} = \sum_{j,k=1}^{22} [\mathcal{B}_j^{in} - \mathcal{B}_j(f_+)] C_{\mathcal{B}\,jk}^{-1} [\mathcal{B}_k^{in} - \mathcal{B}_k(f_+)]$$

Systematic error:

- In this approach the systematic error is the truncation error of the expansion
- Due to the unitarity constraint, the higher order coefficients cannot grow arbitrarily $\Rightarrow$ the truncation error can be controlled
  - let b_{K+1}^{max} be the maximum value of $|b_{K+1}|$, allowed by the unitarity constraint, for fixed values of b_k, $k \leq K$, obtained from the fit
  - realistic prescription for the error: $\delta f_+(q^2)_{syst} = \frac{b_{K+1}^{max} \, |z^{K+1}|}{1-q^2/m_{B*}^2}$

Systematic error and fitting strategy

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Fitting strategy:

Increase the number of parameters until the systematic error becomes negligible compared to the statistical error along the whole semileptonic region

Optimal fit

j	0	1	2	3
b_j	0.42 ± 0.03	-0.47 ± 0.13	0.2 ± 1.3	-0.8 ± 4.1

Also: $|V_{ub}| = (3.54 \pm 0.30) \times 10^{-3}$ (remarkably close to expectations from global CKM fits)

	total	LCSR	LQCD	Belle	CLEO	Babar-t	BaBar-u
n_{data}	31	1	3+5	3	3	3	12
χ^2	21.0	0	5.1	0.0	2.8	4.3	8.7

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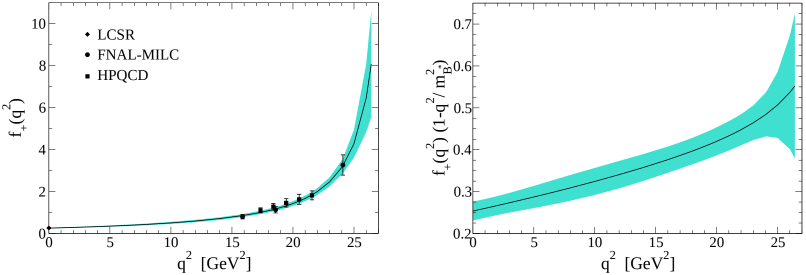

- Errors found by standard $\Delta\chi^2$ analysis, not the linear approximation in the error propagation

Summary and conclusions

- The availability of a reliable bound on an integral involving the square of the modulus of a form factor on the unitarity cut allows one to:
  - constrain the form factor shape parameters at low energy
  - isolate domains in the complex plane where zeros are excluded
  - find bounds at intermediate spacelike regions and test the onset of perturbative QCD
  - control the truncation error of analytic parametrizations
- The knowledge of the phase along a part of the unitarity cut considerably improves the results
- Precise values at points inside the analyticity domain (from ChPT, lattice, experiment) crucial for improving the predictions