Non-leptonic and rare kaon decays in lattice QCD

Chris Sachrajda

School of Physics and Astronomy
University of Southampton
Southampton SO17 1BJ
UK

(RBC-UKQCD Collaboration)

7th International Workshop on Chiral Dynamics,
Jefferson Lab, Newport News
August 6th -10th 2012

UNIVERSITY OF
Southampton
School of Physics
and Astronomy

- At Chiral Dynamics 2009, I presented a talk on *Kaons on the Lattice* in which the main topics were:
    1. Chiral Behaviour
    2. V_{us} from $K_{\ell 2}$ and $K_{\ell 3}$ Decays
    3. The B_K parameter of neutral kaon mixing
    4. $K \to \pi\pi$ Decays

- Recent progress in lattice simulations $\Rightarrow$ quantities such as V_{us} and B_K are known with impressive precision, e.g. using the experimental leptonic and semileptonic widths   FLAG working group, arXiv:1011.4408

$$|V_{us}| = 0.2254(9) \text{ and } |V_{ud}| = 0.97427(21) \quad \text{within the Standard Model}$$

while using also $|V_{ud}| = 0.97425(22)$ J.C.Hardy and I.S.Towner, arXiv:0812.1202

$$
\begin{aligned}
|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 &= 1.0000(7) \quad \text{using lattice semileptonic form factor } f_+(0) \\
&= 0.9999(6) \quad \text{using lattice ratio of decay constants } f_K/f_\pi
\end{aligned}
$$

- For B_K FLAG quotes:

$$\hat{B}_K = 0.738(20).$$

- For $K \to \pi\pi$ decays I discussed the prospects for the evaluation of the amplitudes.

- At Chiral Dynamics 2012, I continue the story. As the precision of lattice calculations continues to improve, it becomes both possible and necessary to extend the range of physical quantities being studied.

- Outline of Talk:

2. $K \to (\pi\pi)_{I=2}$ decay amplitudes

T. Blum, P.A. Boyle, N.H. Christ, N. Garron, E. Goode, T. Izubuchi, C. Jung, C. Kelly, C. Lehner, M. Lightman, Q. Liu, A.T. Lytle, R.D. Mawhinney, C.T. Sachrajda, A. Soni, C. Sturm,   arXiv:1111.1699, arXiv1206.5142.

- In this section I demonstrate that we can calculate $\mathrm{Re}\,A_2$ and $\mathrm{Im}\,A_2$ and to present our results.

- Of course we would also like to determine the $K \to (\pi\pi)_{I=0}$ amplitude A_0 and to understand the $\Delta I = 1/2$ rule and the value of ε'/ε and I review our significant progress towards achieving this in the following section.

- In the meantime however, we know $\mathrm{Re}\,A_0$ and $\mathrm{Re}\,A_2$ from experiment.
  The experimental value of ε'/ε gives us one relation between $\mathrm{Im}\,A_0$ and $\mathrm{Im}\,A_2$, thus if we evaluate $\mathrm{Im}\,A_2$ then within the standard model we know $\mathrm{Im}\,A_0$ to some precision.   Thanks to Andrzej Buras for stressing this to me.

  - I stress again that soon of course, we wish to do better than this.

- Before discussing the direct evaluation of $K \to (\pi\pi)_{I=2}$ decay amplitudes, I need to introduce the ensembles we use.

We have three datasets of $N_f = 2 + 1$ DWF with the Iwasaki Gauge Action:

1 $a \simeq 0.114\,\text{fm}$ $\quad \{24^3 \times 64 \times 16\ (L \simeq 2.74\,\text{fm}) \text{ and } 16^3 \times 32 \times 16\ (L \simeq 1.83\,\text{fm})\}$

arXiv:0804.0473, hep-lat/0701013

- Four light-quark masses corresponding to $m_\pi \simeq 330, 415, 555,$ and $670\,\text{MeV}$.
- The lightest partially quenched pion has a mass of about $240\,\text{MeV}$.
- Only data from masses with $m_\pi \lesssim 420\,\text{MeV}$ are used in the analyses.

2 $a \simeq 0.086\,\text{fm}$ $\quad \{32^3 \times 64 \times 16\quad (L \simeq 2.765\,\text{fm})\}$ $\qquad\qquad$ arXiv:1011.0892

- Three light-quark masses corresponding to $m_\pi \simeq 290, 343$ and $390\,\text{MeV}$
- The lightest partially quenched pion has a mass of about $223\,\text{MeV}$.

For $K \to \pi\pi$ decays we require pions with a physical mass and hence a large volume
$\Rightarrow$ coarse lattice. $\qquad\qquad$ C.Kelly, arXiv:1201.0706; RBC-UKQCD, in preparation

3 $a \simeq 0.14\,\text{fm}$, (DWF+IDSDR) $\quad \{32^3 \times 64 \times 32\ (L \simeq 4.58\,\text{fm})\}$

- Two light-quark masses corresponding to pions with $m_\pi \simeq 170$ and $250\,\text{MeV}$.
- The lightest partially quenched pion has a mass of about $142\,\text{MeV}$.
- The goal was to have a physical $K \to \pi\pi$ decay, with $|p_\pi| = \sqrt{2}\pi/L$.
- With this coarse lattice, it is not surprising that lattice artefacts are the largest source of systematic error. We mitigate against this in a number of ways.

- The operators whose matrix elements have to be calculated are:

$$
\begin{aligned}
O_{(27,1)}^{3/2} &= (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L \\
O_7^{3/2} &= (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_R - (\bar{d}^j d^j)_R \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_R \\
O_8^{3/2} &= (\bar{s}^i d^j)_L \left\{ (\bar{u}^j u^i)_R - (\bar{d}^j d^i)_R \right\} + (\bar{s}^i u^j)_L (\bar{u}^j d^i)_R
\end{aligned}
$$

- It is convenient to use the Wigner-Eckart Theorem: (Notation - $O_{\Delta I_z}^{\Delta I}$)

$$
_{I=2}\langle \pi^+(p_1)\pi^0(p_2) | O_{1/2}^{3/2} | K^+ \rangle = \frac{\sqrt{3}}{2} \langle \pi^+(p_1)\pi^+(p_2) | O_{3/2}^{3/2} | K^+ \rangle,
$$

where

- $O_{3/2}^{3/2}$ has the flavour structure $(\bar{s}d)(\bar{u}d)$.
- $O_{1/2}^{3/2}$ has the flavour structure $(\bar{s}d)((\bar{u}u) - (\bar{d}d)) + (\bar{s}u)(\bar{u}d)$.

- We can then use antiperiodic boundary conditions for the u-quark say, so that the $\pi\pi$ ground-state is $\langle \pi^+(\pi/L)\pi^+(-\pi/L)|$. C-h Kim, Ph.D. Thesis
  - • Do not have to isolate an excited state. •
  - Size (L) needed for physical $K \to \pi\pi$ decay halved.

units	m_π	m_K	$E_{\pi,2}$	$E_{\pi\pi,0}$	$E_{\pi\pi,2}$	$m_K - E_{\pi\pi,2}$
lattice	0.1042(2)	0.3707(7)	0.1739(9)	0.2100(4)	0.356(2)	0.015(2)
MeV	142.1(9)	505.5(3.4)	237(2)	286(2)	486(4)	20.0(3.1)

- The subscripts 0 and 2 refer to $|p_\pi| = 0$ and $\sqrt{2}\pi/L$ respectively.
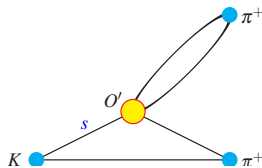
Source	$\mathrm{Re}(A_2)$ $(10^{-8}\,\mathrm{GeV})$
$t_K = 20$	1.41(6)
$t_K = 24$	1.35(6)
$t_K = 28$	1.43(7)
$t_K = 32$	1.30(9)
Weighted Average	1.38(4)
Experiment	1.5

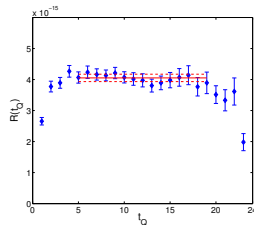Stat. error only

$$O'^{3/2}_{(27,1)} = (\bar{s}d)_L\,(\bar{u}d)_L$$

$$O'^{3/2}_7 = (\bar{s}d)_L\,(\bar{u}d)_R$$
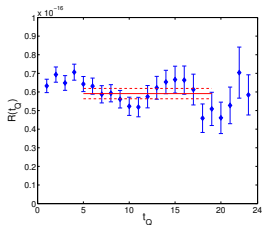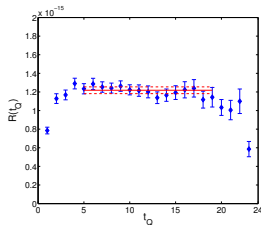
$$O'^{3/2}_8 = (\bar{s}^i d^j)_L\,(\bar{u}^j d^i)_R$$

Sample plateaus for the matrix elements at matched kinematics ($p_\pi = \sqrt{2}p_{\min}$).

	$\mathrm{Re}A_2$	$\mathrm{Im}A_2$
lattice artefacts	15%	15%
finite-volume corrections	6.0%	6.5%
partial quenching	3.5%	1.7%
renormalization	1.8%	5.6%
unphysical kinematics	0.4%	0.8%
derivative of the phase shift	0.97%	0.97%
Wilson coefficients	6.6%	6.6%
Total	18%	19%

- The dominant error is due to lattice artefacts and the fact that out lattice is coarse. This will be eliminated when the calculation is repeated at a second lattice spacing.

- The 15% estimate, intended to be conservative, is obtained by

    1. Studying the dependence on a of quantities which have been calculated at several lattice spacings.
    2. In particular by determining the a dependence of B_K, which is also given by the matrix element of a (27,1) operator.

Our results for the amplitude A_2 are:

$$\mathrm{Re}\,A_2 = (1.381 \pm 0.046_{\mathrm{stat}} \pm 0.258_{\mathrm{syst}})\,10^{-8}\,\mathrm{GeV}$$
$$\mathrm{Im}\,A_2 = -(6.54 \pm 0.46_{\mathrm{stat}} \pm 1.20_{\mathrm{syst}})\,10^{-13}\,\mathrm{GeV}.$$

- The result for $\mathrm{Re}\,A_2$ agrees well with the experimental value of $1.479(4) \times 10^{-8}\,\mathrm{GeV}$ obtained from K^+ decays and $1.573(57) \times 10^{-8}\,\mathrm{GeV}$ obtained from K_S decays .
- $\mathrm{Im}\,A_2$ is unknown so that our result provides its first direct determination.
- For the phase of A_2 we find $\mathrm{Im}\,A_2/\mathrm{Re}\,A_2 = -4.42(31)_{\mathrm{stat}}(89)_{\mathrm{syst}}\,10^{-5}$.
- Combining our result for $\mathrm{Im}\,A_2$ with the experimental results for $\mathrm{Re}\,A_2$, $\mathrm{Re}\,A_0 = 3.3201(18)\dot{1}0^{-7}$ GeV and ε'/ε we obtain:

$$\frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0} = -1.61(19)_{\mathrm{stat}}(20)_{\mathrm{syst}} \times 10^{-4}\,.$$

(Of course, we wish to confirm this directly.)

$$\frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0} = \frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\sqrt{2}\,|\varepsilon|}{\omega}\frac{\varepsilon'}{\varepsilon}$$

$$-1.61(19)_{\mathrm{stat}}(20)_{\mathrm{syst}} \times 10^{-4} = -4.42(31)_{\mathrm{stat}}(89)_{\mathrm{syst}} \times 10^{-5} - 1.16(18) \times 10^{-4}\,.$$

Writing A_2 in terms of the matrix elements for the physical $K^+ \to \pi^+ \pi^0$ decay:

$$A_2 = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{1}{\sqrt{3}} \sum_i C_i(3 \text{ GeV}) \mathscr{A}'^{\,\overline{\text{MS}}\text{-NDR}}_i(3 \text{ GeV}).$$

we find the matrix elements to be

$$\begin{aligned}
\mathscr{A}'^{\,\overline{\text{MS}}\text{-NDR}}_{(27,1)}(3 \text{ GeV}) &= 0.0461(14) \text{ GeV}^3 \\
\mathscr{A}'^{\,\overline{\text{MS}}\text{-NDR}}_{(8,8)}(3 \text{ GeV}) &= 0.874(49) \text{ GeV}^3 \\
\mathscr{A}'^{\,\overline{\text{MS}}\text{-NDR}}_{(8,8)_{\text{mix}}}(3 \text{ GeV}) &= 3.96(23) \text{ GeV}^3.
\end{aligned}$$

Their contributions to A_2 are:

$\text{Re}(A_2)_{(27,1)}$	$(1.398 \pm 0.044)\,10^{-8}\,\text{GeV}$	$\text{Im}(A_2)_{(27,1)}$	$(1.55 \pm 0.36)\,10^{-13}\,\text{GeV}$
$\text{Re}(A_2)_{(8,8)}$	$(4.29 \pm 0.24)\,10^{-11}\,\text{GeV}$	$\text{Im}(A_2)_{(8,8)}$	$(4.47 \pm 0.25)\,10^{-14}\,\text{GeV}$
$\text{Re}(A_2)_{(8,8)_{\text{mix}}}$	$(-2.14 \pm 0.12)\,10^{-10}\,\text{GeV}$	$\text{Im}(A_2)_{(8,8)_{\text{mix}}}$	$(-8.14 \pm 0.47)\,10^{-13}\,\text{GeV}.$

- The *ab initio* calculation of A_2 described above builds upon substantial theoretical advances, achieved over many years.
- The agreement we find for $\mathrm{Re}\, A_2$ with the experimental result is very satisfying.
- We are also able to determine $\mathrm{Im}\, A_2$ for the first time.
- It will be important to repeat this calculation using a second lattice spacing so that a continuum extrapolation can be performed thus eliminating the dominant contribution to the error, reducing the total uncertainty to about 5%.
- We expect that the dominant remaining errors in A_2 will then come from the omission of electromagnetic and other isospin breaking mixing between the large amplitude A_0 and A_2.
- We now turn to the evaluation of the amplitude A_0.

The 2012 KWLA panel is proud to award

The 2012 Ken Wilson Lattice Award

To:

T. Blum   C. Jung   R.D.Mawhinney
P.A. Boyle   C. Kelly   C.T. Sachrajda
N.H. Christ   C. Lehner   A. Soni
N. Garron   M. Lightman   C. Sturm
E. Goode   Q. Liu
T. Izubuchi   A.T. Lytle

In recognition of their paper titled

$K \to (\pi\pi)_{I=2}$ Decay Amplitude from Lattice QCD

The 2012 KWLA Panel Members
S. Aoki, W. Detmold, G. Fleming, D. Lin, H. Meyer, J. Zanotti

- For this work we received the 2012 Ken Wilson Lattice award at Lattice 2012.
- Criteria: The paper must be important research beyond the existing state of the art. $\cdots$

T. Blum, P.A. Boyle, N.H. Christ, N. Garron, E. Goode, T. Izubuchi, C. Lehner, Q. Liu, R.D. Mawhinney,
C.T. Sachrajda, A. Soni, C. Sturm, H. Yin, R. Zhou

- The $I = 0$ final state has vacuum quantum numbers.
- Vacuum contribution must be subtracted; disconnected diagrams require statistical cancelations to obtain the $e^{-2m_\pi t}$ behaviour.
- Consider first the two-pion correlation functions, which are an important ingredient in the evaluation of $K \rightarrow \pi\pi$ amplitudes.
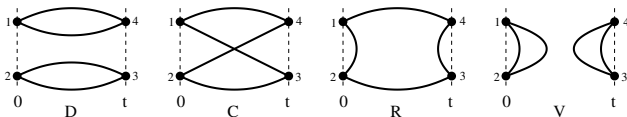
- For I=2 $\pi\pi$ states the correlation function is proportional to D-C.
- For I=0 $\pi\pi$ states the correlation function is proportional to 2D+C-6R+3V.

 The major practical difficulty is to subtract the vacuum contribution with sufficient precision.

- In the paper we report on high-statistics experiments on a $16^3 \times 32$ lattice, $a^{-1} = 1.73$ GeV, $m_\pi = 420$ MeV, with the propagators evaluated from each time-slice.

Diagrams contributing to two-pion correlation functions
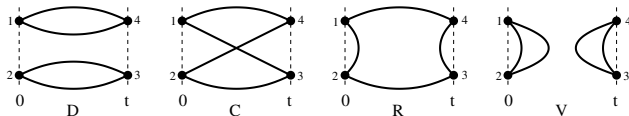
- For I=2 $\pi\pi$ states the correlation function is proportional to D-C.
- For I=0 $\pi\pi$ states the correlation function is proportional to 2D+C-6R+3V.

RBC/UKQCD, Qi Liu – Lattice 2010
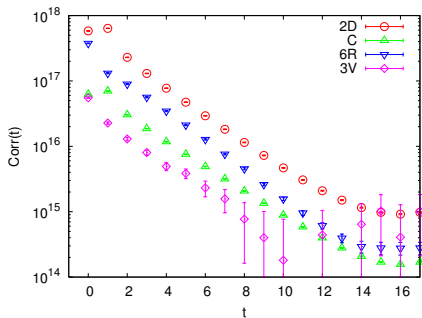
Two-pion Correlation Functions

RBC/UKQCD, arXiv:1106.2714
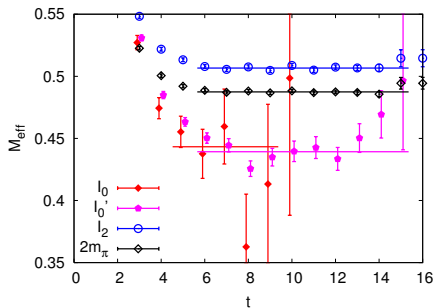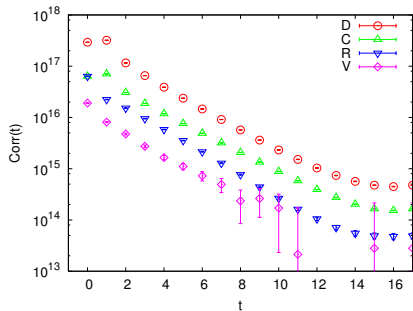
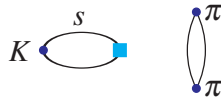- $M_{\text{eff}} = \log C(t)/C(t+1)$.
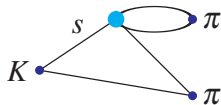
Type1
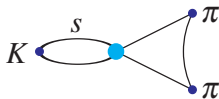
Type2
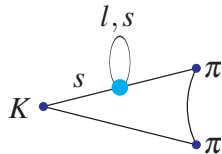
Type3
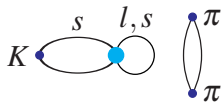
Type4
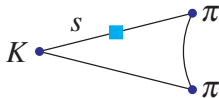
Mix3

Mix4

- There are 48 different contractions and we classify the contributions into the 6 different types illustrated above.
- Mix3 and Mix4 are needed to subtract the power divergences which are proportional to matrix elements of $\bar{s}\gamma_5 d$.

- These results are for the $K \to \pi\pi$ (almost) on-shell amplitudes with 420 MeV pions at rest:

$$
\begin{array}{ll}
\mathrm{Re}\, A_0 & (3.80 \pm 0.82)\, 10^{-7}\ \mathrm{GeV} \\
\mathrm{Im}\, A_0 & -(2.5 \pm 2.2)\, 10^{-11}\ \mathrm{GeV} \\
\mathrm{Re}\, A_2 & (4.911 \pm 0.031)\, 10^{-8}\ \mathrm{GeV} \\
\mathrm{Im}\, A_2 & -(5.502 \pm 0.0040)\, 10^{-13}\ \mathrm{GeV}
\end{array}
$$

- This is an exploratory exercise in which we are learning how to do the calculation.

- Since this work was finished we have been developing techniques which seem to enhance the signal considerably.

- The exploratory results for $K \to (\pi\pi)_{I=0}$ decays encourage us to proceed to physical kinematics.

  $\Rightarrow$ an understanding of the $\Delta I = 1/2$ rule and the value of ε'/ε.

- The evaluation of disconnected diagram has allowed us to study the η and η' mesons and their mixing.

- In his thesis Qi Liu extended the above study to the $24^3 \times 64$ ensembles.

 1. $16^3 \times 32$ ensembles; 877 MeV kaon decaying into two 422 MeV pions at rest:

  $$\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} = 9.1\,(21)\,.$$

  2. $24^3 \times 64$ ensembles; 662 MeV kaon decaying into two 329 MeV pions at rest:

  $$\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} = 12.0\,(17)\,.$$

- Whilst both these results are obtained at unphysical kinematics and are different from the physical value of 22.5, it is nevertheless interesting to understand the origin of these enhancements.

- 99% of the contribution to the real part of A_0 and A_2 come from the matrix elements of the current-current operators.
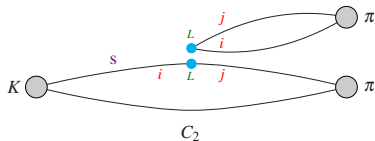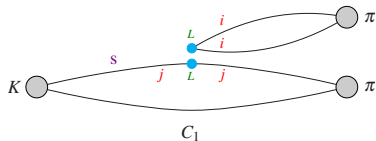
- We find that approximately

$$\mathrm{Re}\, A_2 \propto \sqrt{\frac{2}{3}}\,(C_1 + C_2) \quad \text{and} \quad \mathrm{Re}\, A_0 \propto \sqrt{\frac{1}{3}}\,(2C_1 - C_2).$$

- Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$ so that the two amplitudes are approximately equal.
- We find instead that $C_2 \approx -C_1$ so that A_2 is significantly suppressed!
- A_2 has a larger kinematic dependence than A_0.
- We are now preparing this calculation for publication and moving on to the next generation calculation with advanced techniques such as G-parity boundary conditions, all-to-all propagators etc.

• Extending the reach of Lattice Simulations •

- More recently we have began to consider long-distance contributions to physical quantities. These are not given in terms of matrix elements of local operators, but require the evaluation for example of:

$$\int d^4x \int d^4y \, \langle h_2 \, | \, T\{O_1(x)\,O_2(y)\} \, | \, h_1 \rangle .$$

- The most advanced of our projects is on the evaluation of long-distance contributions to the $K_L - K_S$ mass difference.

  Jianglei Yu, arXiv:1111.6953; paper in preparation.

$$\int d^4x \int d^4y \, \langle \bar{K}^0 \, | \, T\{H_W(x)\,H_W(y)\} \, | \, K^0 \rangle .$$

- In the following section I will present some preliminary thoughts about the rare kaon decays $K \to \pi \nu \bar{\nu}$ and $K \to \pi \ell^+ \ell^-$:

$$\int d^4x \, e^{-iq\cdot x} \int d^4y \, \langle \pi \, | \, T\{J^\mu(x)\,H_W(y)\} \, | \, K^0 \rangle .$$

- Up to now, the main theoretical tool for these processes has been Chiral Perturbation Theory with its many limitations and uncertainties.
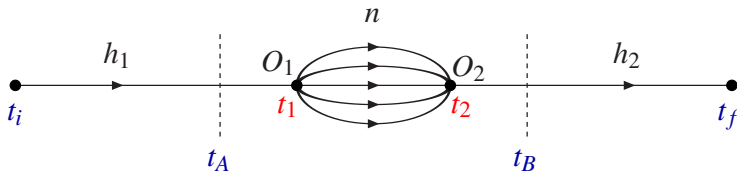
- How do you prepare the states $h_{1,2}$ in

$$\int d^4x \int d^4y \, \langle h_2 \,|\, T\{O_1(x)\,O_2(y)\} \,|\, h_1 \rangle,$$

when the time of the operators is integrated.

- The practical solution is to integrate over a large subinterval in time $t_A \leq t_{x,y} \leq t_B$, but to create h_1 and to annihilate h_2 well outside of this region:

- This is the natural modification of standard field theory for which the asymptotic states are prepared at $t \to \pm\infty$ and then the operators are integrated over all time.

- This approach has been successfully implemented in the ΔM_K project.

  N.Christ arXiv:1012.6034; Jianglei Yu arXiv:1111.6953; paper in preparation
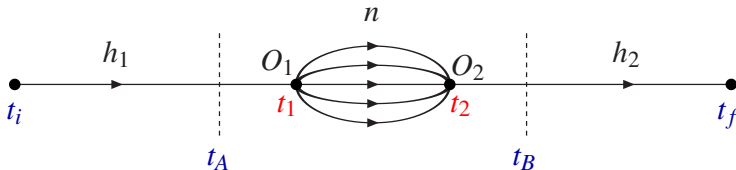
- Δm_K is given by

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = \frac{1}{2m_K} 2\mathscr{P} \sum_\alpha \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, \alpha \rangle \langle \alpha \,|\, \mathscr{H}_W \,|\, K^0 \rangle}{m_K - E_\alpha} = 3.483(6) \times 10^{-12}\,\text{MeV}.$$

- The above correlation function gives ($T = t_B - t_A + 1$)

$$C_4(t_A, t_B; t_i, t_f) = |Z_K|^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, n \rangle \langle n \,|\, \mathscr{H}_W \,|\, K_0 \rangle}{(m_K - E_n)^2} \times$$
$$\left\{ e^{(M_K - E_n)T} - (m_K - E_n)T - 1 \right\}.$$

- From the coefficient of T we can, (in principle at least), obtain

$$\Delta m_K^{\text{FV}} \equiv 2 \sum_n \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, n \rangle \langle n \,|\, \mathscr{H}_W \,|\, K_0 \rangle}{(m_K - E_n)}.$$

In order to evaluate Δm_K we need to be able to:

- Relate Δm_K and Δm_K^{FV}. ✓   RBC-UKQCD; N.H.Christ, G.Martinelli, CTS (in preparation)
  This is a significant extension of the theory of finite-volume effects for two-pion states: the Lüscher quantization condition, Lellouch-Lüscher factor, $\cdots$.

- Control the additional ultraviolet divergences when the weak Hamiltonians are close together. ✓   J.Yu, arXiv:1111.6953; RBC-UKQCD (in preparation)
  This is facilitated by the GIM mechanism which requires the presence of charm quarks.

- $\Delta S = 1$ effective weak Hamiltonian including four flavours:
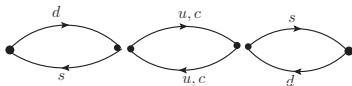
$$H_W = \frac{G_F}{\sqrt{2}} \sum_{q,q'=u,c} V_{qd} V_{q's}^* (C_1 Q_1^{qq'} + C_2 Q_2^{qq'})$$
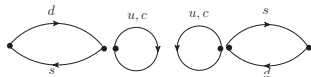
where

$$Q_1^{qq'} = (\bar{s}_i d_i)_{V-A} (\bar{q}_j q'_j)_{V-A} \quad \text{and} \quad Q_2^{qq'} = (\bar{s}_i d_j)_{V-A} (\bar{q}_j q'_i)_{V-A}.$$
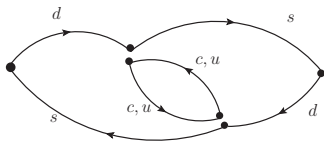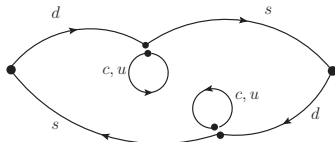
- Evaluation of graphs such as:

Type 1

Type 3

Type 2

Type 4

- In our exploratory study on the 16^3 ensembles with $m_\pi = 420$ MeV, we only evaluate Type 1 and Type 2 graphs.
- We obtain Δm_K in the range $\{5.81(28) - 10.58(75)\} \times 10^{-12}$ MeV as m_K is varied from 563 to 839 MeV. (The physical value is $3.483(6) \times 10^{-12}$ MeV.)

- As an example of our investigations consider the behaviour of the integrated $Q_1 - Q_1$ correlation function without GIM subtraction but with an artificial cut-off, $R = \sqrt{\{(t_2 - t_1)^2 + (\vec{x}_2 - \vec{x}_1)^2\}}$ on the coordinates of the two Q_1 insertions.
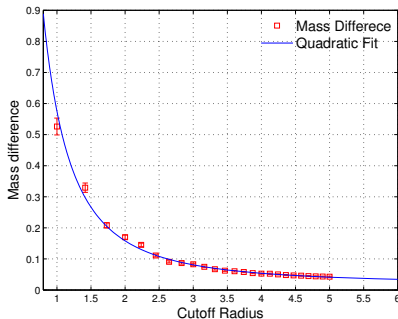
- The plot exhibits the quadratic divergence as the two operators come together.
- The quadratic divergence is cancelled by the GIM mechanism.

F.Mescia, C,Smith, S.Trine hep-ph/0606081

- Rare kaon decays which are dominated by short-distance FCNC processes, $K \to \pi \nu \bar\nu$ in particular, provide a potentially valuable window on new physics at high-energy scales.

- The decays $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$ are also considered promising because the long-distance effects are reasonably under control using ChPT.

  - They are sensitive to different combinations of short-distance FCNC effects and hence in principle provide additional discrimination to the neutrino modes.
  - A challenge for the lattice community is therefore either to calculate the long-distance effects reliably or at least to determine the Low Energy Constants of ChPT.

$K_L \rightarrow \pi^0 \ell^+ \ell^-$ **cont.**

There are three main contributions to the amplitude:

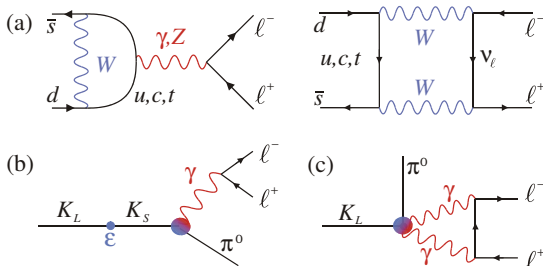**1** Short distance contributions: F.Mescia, C,Smith, S.Trine hep-ph/0606081

$$H_{\text{eff}} = -\frac{G_F \alpha}{\sqrt{2}} V_{ts}^* V_{td} \{ y_{7V} (\bar{s}\gamma_\mu d) (\bar{\ell}\gamma^\mu \ell) + y_{7A} (\bar{s}\gamma_\mu d) (\bar{\ell}\gamma^\mu \gamma_5 \ell) \} + \text{h.c.}$$

- Direct CP-violating contribution.
- In BSM theories other effective interactions are possible.

2 Long-distance indirect CP-violating contribution

$$A_{ICPV}(K_L \rightarrow \pi^0 \ell^+ \ell^-) = \varepsilon A(K_1 \rightarrow \pi^0 \ell^+ \ell^-).$$

3 The two-photon CP-conserving contribution $K_L \rightarrow \pi^0 (\gamma^* \gamma^* \rightarrow \ell^+ \ell^-)$.

- The current phenomenological status for the SM predictions is nicely summarised by: V.Cirigliano et al., arXiv1107.6001

$$\text{Br}(K_L \to \pi^0 e^+ e^-)_{\text{CPV}} = 10^{-12} \times \left\{ 15.7|a_S|^2 \pm 6.2|a_S| \left(\frac{\text{Im}\,\lambda_t}{10^{-4}} \right) + 2.4 \left(\frac{\text{Im}\,\lambda_t}{10^{-4}} \right)^2 \right\}$$

$$\text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPV}} = 10^{-12} \times \left\{ 3.7|a_S|^2 \pm 1.6|a_S| \left(\frac{\text{Im}\,\lambda_t}{10^{-4}} \right) + 1.0 \left(\frac{\text{Im}\,\lambda_t}{10^{-4}} \right)^2 \right\}$$

- $\lambda_t = V_{td}V_{ts}^*$ and $\text{Im}\,\lambda_t \simeq 1.35 \times 10^{-4}$.
- $|a_S|$, the amplitude for $K_S \to \pi^0 \ell^+ \ell^-$ at $q^2 = 0$ as defined below, is expected to be $O(1)$ but the sign of a_S is unknown. $|a_S| = 1.06^{+0.26}_{-0.21}$.
- For $\ell = e$ the two-photon contribution is negligible.
- Taking the positive sign (?) the prediction is

$$\begin{aligned}
\text{Br}(K_L \to \pi^0 e^+ e^-)_{\text{CPV}} &= (3.1 \pm 0.9) \times 10^{-11} \\
\text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPV}} &= (1.4 \pm 0.5) \times 10^{-11} \\
\text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPC}} &= (5.2 \pm 1.6) \times 10^{-12}.
\end{aligned}$$

- The current experimental limits (KTeV) are:

$$\text{Br}(K_L \to \pi^0 e^+ e^-) < 2.8 \times 10^{-10} \quad \text{and} \quad \text{Br}(K_L \to \pi^0 \mu^+ \mu^-) < 3.8 \times 10^{-10}.$$

G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- We now turn to the CPC decays $K_S \to \pi^0 \ell^+ \ell^-$ and $K^+ \to \pi^+ \ell^+ \ell^-$ and consider

$$T_i^\mu = \int d^4 x \, e^{-iq \cdot x} \langle \pi(p) | \, \mathrm{T}\{J_{\mathrm{em}}^\mu(x)\, Q_i(0)\,\} \, | K(k) \rangle \,,$$

  where Q_i is an operator from the effective Hamiltonian.

- Gauge invariance implies that

$$T_i^\mu = \frac{\omega_i(q^2)}{(4\pi)^2} \left\{ q^2(p+k)^\mu - (m_K^2 - m_\pi^2)\, q^\mu \right\} \,.$$

- Within ChPT the Low energy constants a_+ and a_S are defined by

$$a = \frac{1}{\sqrt{2}} V_{us}^* V_{ud} \left\{ C_1 \omega_1(0) + C_2 \omega_2(0) + \frac{2N}{\sin^2 \theta_W} f_+(0) C_{7V} \right\}$$

  where $Q_{1,2}$ are the two current-current GIM subtracted operators and the C_i are
  the Wilson coefficients. (C_{7V} is proportional to y_{7V} above).

G.Ambosio, G.Ecker, G.Isidori and J.Portoles, hep-ph/9808289

- Phenomenological values: $a_+ = -0.578 \pm 0.016$ and $|a_S| = 1.06^{+0.26}_{-0.21}$.

Can we do better in lattice simulations?

- The generic non-local matrix elements which we wish to evaluate is

$$
\begin{aligned}
X &\equiv \int_{-\infty}^{\infty} dt_x \, d^3x \, \langle \pi(p) | \mathrm{T} \left[J(0) H(x) \right] | K \rangle \\
&= i \sum_n \frac{\langle \pi(p) | J(0) | n \rangle \langle n | H(0) | K \rangle}{m_K - E_n + i\varepsilon} - i \sum_{n_s} \frac{\langle \pi(p) | H(0) | n_s \rangle \langle n_s | J(0) | K \rangle}{E_{n_s} - E_\pi + i\varepsilon} \, ,
\end{aligned}
$$

  - $\{|n\rangle\}$ and $\{|n_s\rangle\}$ represent complete sets of non-strange and strange sets.

- In Euclidean space we envisage calculating correlation functions of the form

$$
C \equiv \int_{-T_a}^{T_b} dt_x \, \langle \phi_\pi(\vec{p}, t_\pi) \mathrm{T} \left[J(0) H(t_x) \right] \phi_K^\dagger(t_K) \rangle \equiv \sqrt{Z_K} \, \frac{e^{-m_K |t_K|}}{2m_K} \, X_E \, \sqrt{Z_\pi} \, \frac{e^{-E_\pi t_\pi}}{2E_\pi} \, ,
$$

where

$$
\begin{aligned}
X_{E_-} &= -\sum_n \frac{\langle \pi(p) | J(0) | n \rangle \langle n | H(0) | K \rangle}{m_K - E_n} \left(1 - e^{(m_K - E_n) T_a} \right) \quad \text{and} \\
X_{E_+} &= \sum_{n_s} \frac{\langle \pi(p) | H(0) | n_s \rangle \langle n_s | J(0) | K \rangle}{E_{n_s} - E_\pi} \left(1 - e^{-(E_{n_s} - E_\pi) T_b} \right) .
\end{aligned}
$$

- Chiral ward identities imply that we can add a term proportional to the scalar density $\bar{s}d$ to the Hamiltonian without changing physical results. We can therefore subtract the single pion intermediate state by imposing $\langle\pi|H+c_S\,\bar{s}d|K\rangle=0$.

- It is instructive to see how this works in the present case at lowest order in chiral perturbation theory. The scalar density in the effective theory can be written as

$$S^{sd}=\mathrm{Tr}\left[\lambda^{sd}\left(\Sigma+\Sigma^\dagger\right)\right]\quad\text{where}\quad\lambda^{sd}=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}.$$

- The em current is of the form

$$J^\mu=i\frac{f^2}{4}\,\mathrm{Tr}\left[Q\left(\Sigma\partial^\mu\Sigma^\dagger+\Sigma\partial^\mu\Sigma\right)\right]$$
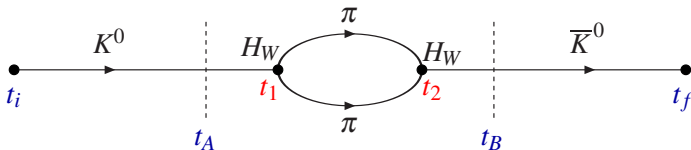
- The c_S term leads to additional diagrams:

which are proportional to

$$\frac{(p_\pi+p_K)^\mu}{p_K^2-m_\pi^2}+\frac{(p_\pi+p_K)^\mu}{p_\pi^2-m_K^2}\,.$$

On shell, when $p_K^2=m_K^2$ and $p_\pi^2=m_\pi^2$, the sum of the two terms indeed gives zero.

- In the ΔM_K computation, there is, of course, a two-pion intermediate state and we have had to control the corresponding finite-volume effects.
- This has been done on the assumption that the dominant intermediate states below m_K are the two-pion states.

- We have seen that we can remove the single pion intermediate state.
- Which intermediate states contribute?
  - Are there any states below M_K?
  - We can control q^2 and stay below the two-pion threshold.
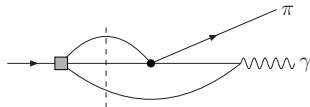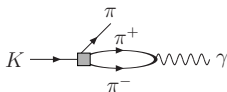
  - Are there two-pion intermediate states as a result of the Wess-Zumino term?
  - Do we need to consider three-pion intermediate states?
  - Answers to the above questions will affect what the finite-volume corrections are?
 - The ChPT-based phemomenology community neglect such possibilities.

All to be investigated further!

- It looks as though the FV corrections are much simpler than for ΔM_K and may be exponentially small?

$$T_i^\mu = \int d^4x \, e^{-iq\cdot x} \langle \pi(p) | \, \mathrm{T}\{J^\mu(x) \, Q_i(0)\} \, | K(k) \rangle,$$

- Each of the two local Q_i operators can be normalized in the standard way and J can be normalized.
- Calculation of long-distance effects $\Rightarrow$ must treat additional divergences as $x \to 0$.
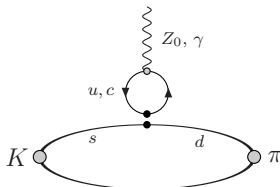

- Quadratic divergence is absent by gauge invariance $\Rightarrow$ Logarithmic divergence.
  - Checked explicitly for Wilson and Clover at one-loop order.

    G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

  - Absence of power divergences does not require GIM.
  - Logarithmic divergence cancelled by GIM.
  - For DWF the same applies for the axial current.
- Control of short-distance effects also appears to be much simpler than for ΔM_K.

To be investigated further!
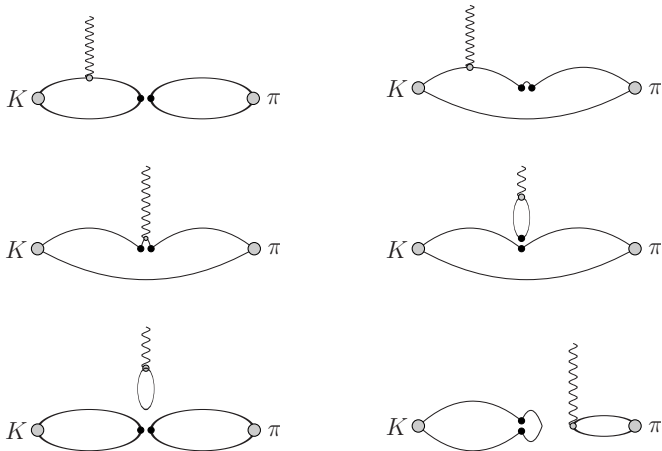
○ Sample diagrams:

+ lots more

○ The last two diagrams are examples of *disconnected* diagrams.

Goal is wide-ranging precision flavour physics

- *Standard* quantities, such as quark masses, decay constants, B_K, formfactors, are now calculated with excellent precision.

- We have performed the first direct calculation of the $K \to (\pi\pi)_{I=2}$ decay amplitude A_2. We believe that this will serve as an important benchmark for future improved calculations.

- Although significant technical problems remain, we are well on our way towards calculating A_0.
  (I did not talk about our exploratory studies refining all-to-all propagators or using G-parity.)

- We are beginning to tackle the calculation of long-distance effects in Δm_K and rare kaon decays.

- As the precision improves we are having to think about electromagnetic and other isospin breaking effects.

- As we extend the range of quantities which are studied in lattice simulations we will need the continued help of the CD community to organise our projects most effectively.