Form Factor of Pion and Lattice QCD

Raul Briceno, Bipasha Chakraborty, Adithia Kusno, *David Richards*

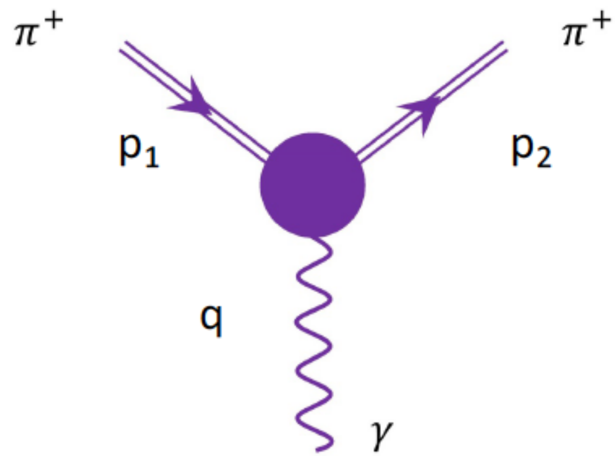
Jefferson Laboratory

25th Sep 2017

Hadron 2017, Salamanca, Spain

Introduction

- Pion is the simplest hadron: near-Goldstone boson.
- No spin, so only a single electric form factor

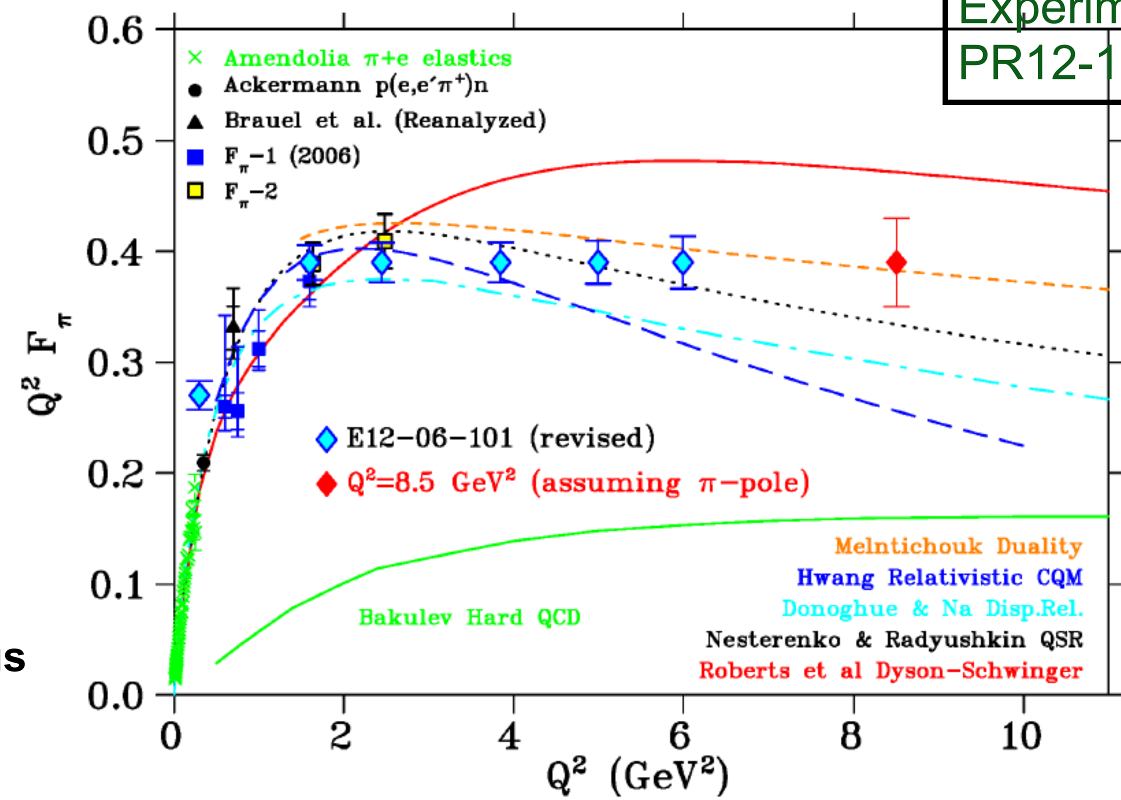


$$\langle \pi^+(\vec{p}\,') | j^\mu | \pi^+(\vec{p}) \rangle = (p + p')^\mu \, F_\pi(Q^2)$$

- Probe transition between hadronic (soft) and quark/ gluon degrees of freedom

Introduction - II

Experimental proposal PR12-16-003



Charge Radius

Approach to quark and gluon degrees of freedom

$$F_\pi(Q^2) \longrightarrow \frac{16\pi\alpha_s(Q^2) f_\pi^2}{Q^2}$$

Lattice QCD

Observables in lattice QCD are then expressed in terms of the path integral as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \prod_{n,\mu} dU_\mu(n) \prod_n d\psi(n) \prod_n d\bar{\psi}(n) \mathcal{O}(U, \psi, \bar{\psi}) e^{-\left(S_G[U] + S_F[U,\psi,\bar{\psi}]\right)}$$

Integrate out the Grassmann variables:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \prod_{n,\mu} dU_\mu(n) \mathcal{O}(U, G[U]) \det M[U] e^{-S_G[U]}$$

Importance Sampling

where $G(U,x,y)^{ij}_{\alpha\beta} \equiv \langle \psi^i_\alpha(x) \bar{\psi}^j_\beta(y) \rangle = M^{-1}(U)$

- Generate an ensemble of gauge configurations

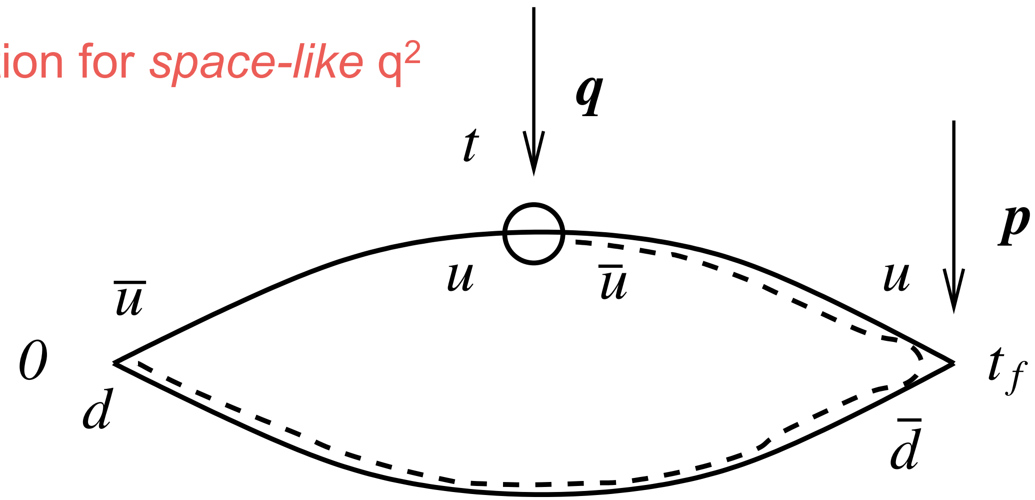
$$P[U] \propto \det M[U] e^{-S_G[U]}$$

- Calculate observable

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}(U^n, G[U^n])$$

Pion Structure

Lattice calculation for *space-like* q[2]



$$\Gamma_{\pi^+\mu\pi^+}(t_f,t;\vec{p},\vec{q}) = \sum_{\vec{x},\vec{y}} \langle 0|\phi(\vec{x},t_f) V_\mu(\vec{y},t)\phi^\dagger(\vec{0},0)|0\rangle e^{-i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}}$$

Resolution of unity – insert states

$$\to \langle 0\,|\phi(0)|\,\pi(\vec{p}_f)\rangle \frac{e^{-(t_f-t)E_\pi(\vec{p}_f)}}{2E_\pi(\vec{p}_f)} \langle \pi(\vec{p}_f)\,|V_\mu(0)|\,\pi(\vec{p}_i)\rangle \frac{e^{-(t-t_i)E_\pi(\vec{p}_i)}}{2E_\pi(\vec{p}_i)} \langle \pi(\vec{p}_i)\,|\phi\dagger(0)|\,0\rangle\,.$$

$$\langle \pi(\vec{p}_f)\,|V_\mu(0)|\,\pi(\vec{p}_i)\rangle_{\mathrm{continuum}} = Z_V \left\langle \pi(\vec{p}_f)\,\middle|V_\mu^{\mathrm{lat}}(0)\middle|\,\pi(\vec{p}_i)\right\rangle = F(Q^2)(p_i+p_f)_\mu$$

Charge conservation

Variational Method

$$C_{2\mathrm{pt}}(t,\vec{p}) = \sum_{\vec{x}} \langle \phi(\vec{x},t)\phi^\dagger(0)\rangle e^{-i\vec{p}\cdot\vec{x}}$$

Signal-to-noise ratio: $C_{2\mathrm{pt}}(t,\vec{p})/C_{\sigma^2}(t) \longrightarrow e^{-((E(\vec{p})-m_\pi)t)}$

Construct matrix of correlators with *judicious choice of operators*

$$C_{ij}(t) = \frac{1}{V_3}\sum_{\vec{x},\vec{y}} \langle \phi_i(\vec{x},t)\phi_j^\dagger(\vec{y},0)\rangle = \sum_N A_N^i A_N^{j\dagger} e^{-E_N t}$$

Delineate contributions using *variational method*: solve

$$C(t)v^{(N)}(t,t_0) = \lambda_N(t,t_0)C(t_0)v^{(N)}(t,t_0).$$

$$\lambda_N(t,t_0) \rightarrow e^{-E_N(t-t_0)}(1+\mathcal{O}(e^{-\Delta E(t-t_0)}))$$

Eigenvectors, with metric $C(t_0)$, are orthonormal and project onto the respective states

$$v^{(N')\dagger}C(t_0)v^{(N)} = \delta_{N,N'}$$

$$Z_i^N = \sqrt{2m_N}e^{m_N t_0/2}v_j^{(N)*}C_{ji}(t_0).$$

Distillation

M. Peardon *et al.*, PRD80,054506 (2009)

Can we evaluate such a matrix efficiently, for reasonable basis of operators?

Introduce $\tilde{\psi}(\vec{x},t) = L(\vec{x},\vec{y})\psi(\vec{y},t)$ **where L is 3D Laplacian**

Write $L \equiv (1 - \kappa\nabla/n)^n = \sum_i f(\lambda_i)\xi^i \times \xi^{*i}$ **where λ_i and ξ_i are eigenvalues and eigenvectors of the Laplacian.**

We now truncate the expansion sufficiently to capture low-energy physics

Insert between each quark field in our correlation function.

Perambulators $\tau^{ij}_{\alpha\beta}(t,0) = \xi^{*i}(t)M^{-1}(t,0)_{\alpha\beta}\xi^j$

$$C_{ij}(t) = \phi_i^{pq}(t)\phi_j^{rs} \times \tau^{pr}(t,0)\tau^{qs\dagger}(t,0)$$

- 3pt functions implemented by replacing one of the perambulators by a so-called generalized perambulator with current inserted.

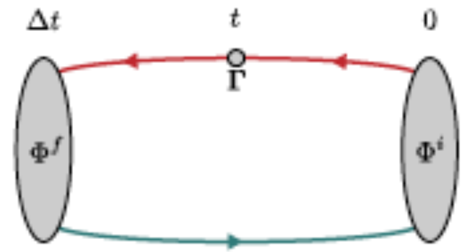
$$S^{ij}(t_f,t,t_i) = \xi^{(i)\dagger}(t_f)M^{-1}(t_f,t)\Gamma(t)M^{-1}(t,t_i)\xi^{(j)}(t_i)$$

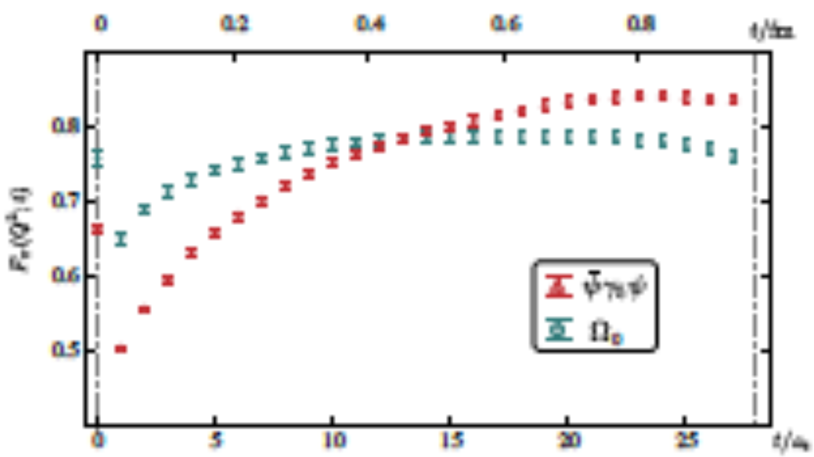
Distillation for Pion Form Factor

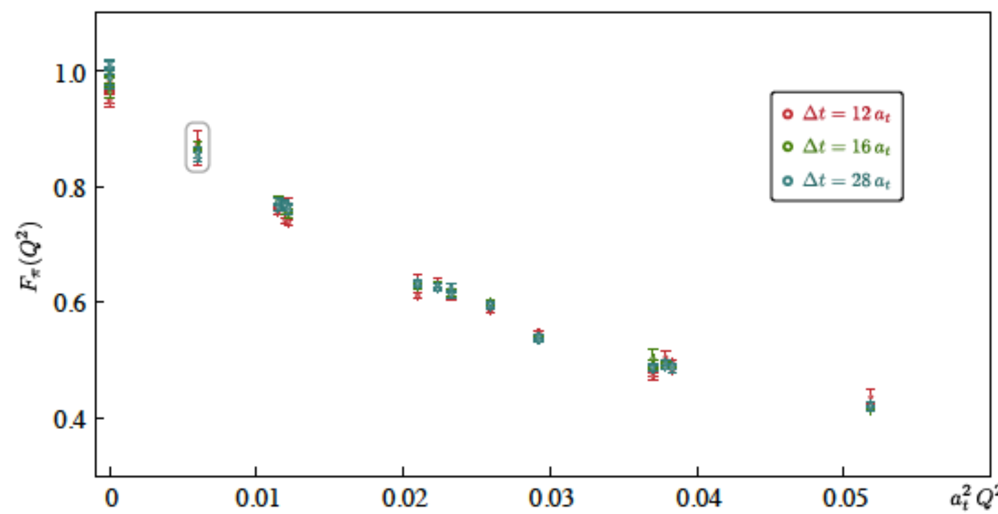
- Formalism for EM matrix elements already demonstrated for mesons by HadSpec collaboration.

$$F(Q^2;t) = F(Q^2) + f_f e^{-\delta E_f(\Delta t - t)} + f_i e^{-\delta E_i t}$$

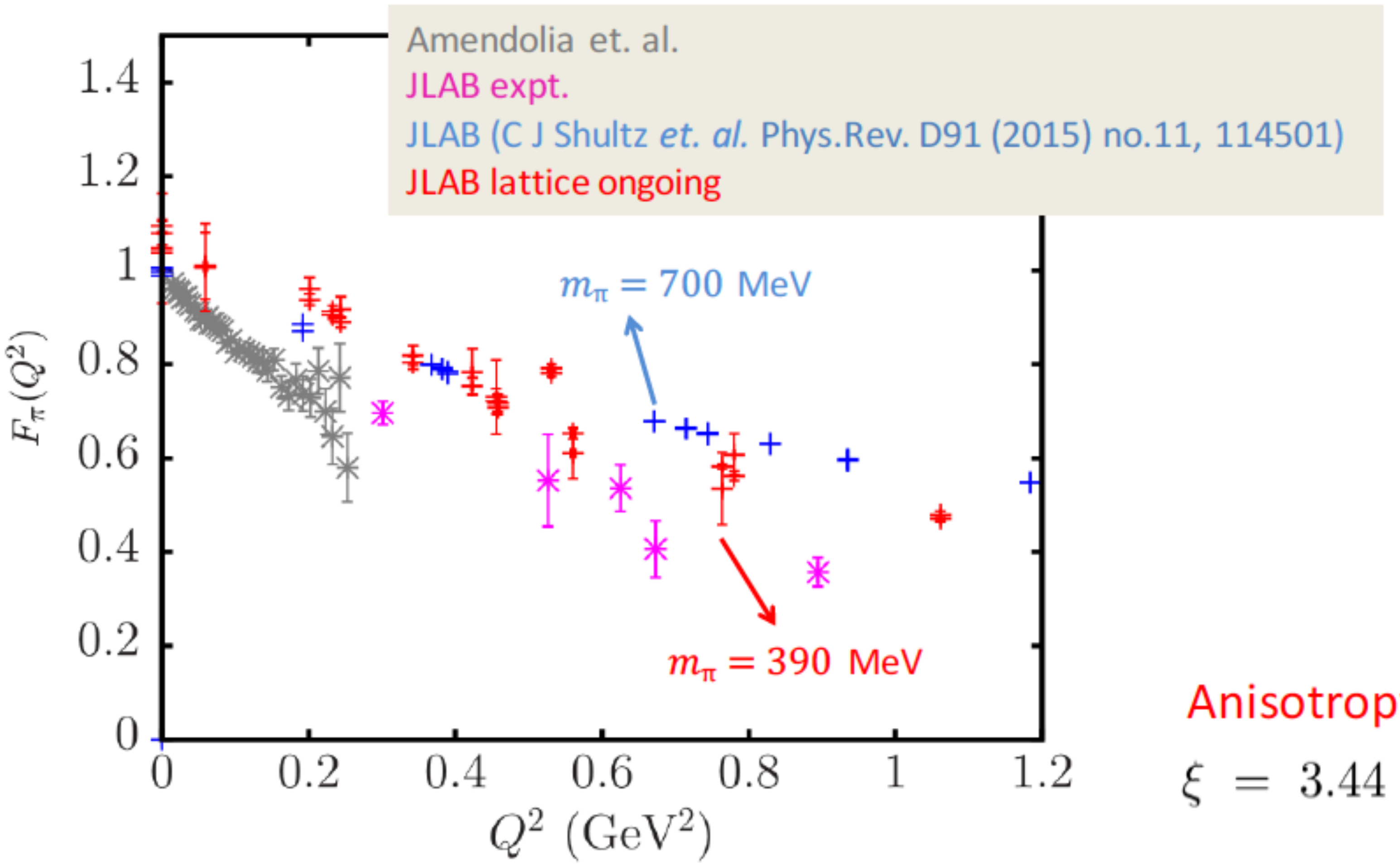
Shultz, Dudek and Edwards, arXiv:1501.07457







Distillation for Pion Form Factor - II



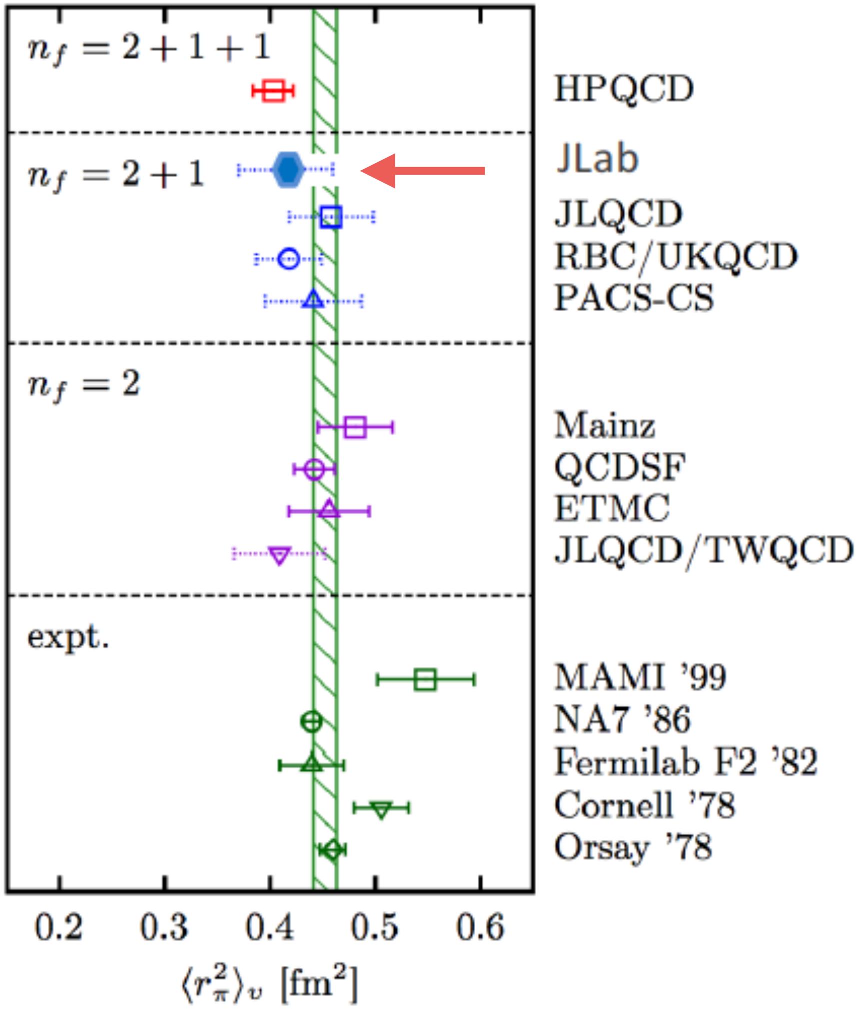
Anisotropy

$$\xi = 3.44$$

$$(L/a_s)^3 \times (T/a_t) = 20^3 \times 128$$

$$a_s = 0.12 \text{ fm}, \quad \frac{a_s}{a_t} = 3.44$$

Charge Radius

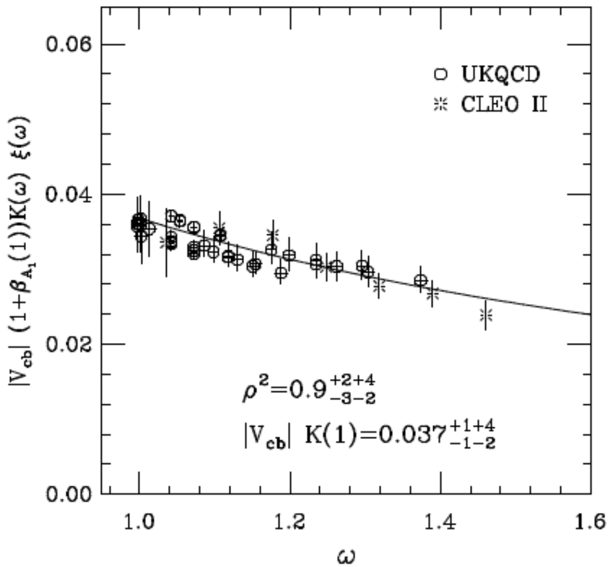


HPQCD, PRD93, 054503 (2017)

Slope of form factor at $Q^2 = 0$ gives charge radius

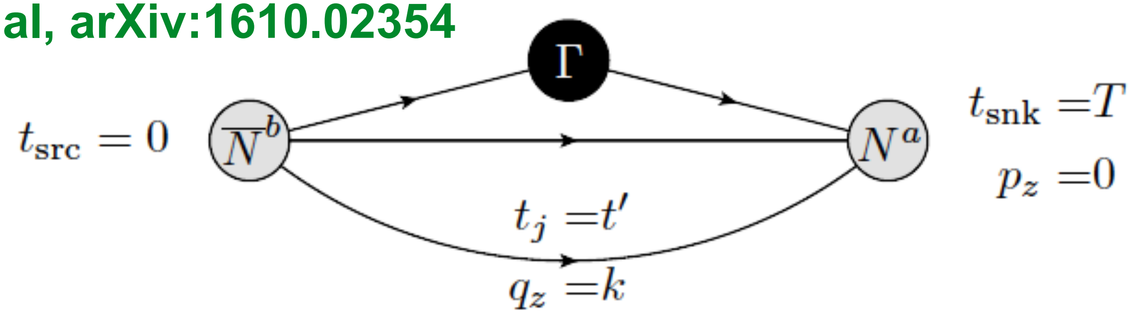
Calculate slope at zero recoil..

UKQCD, L. Lellouch et al., Nucl. Phys. B444, 401 (1995), hep-lat/9410013



Moment Methods

Bouchard et al, arXiv:1610.02354



- Introduce three-momentum projected three-point function

$$C^{3\rm pt}(t,t') = \sum_{\vec{x},\vec{x}'} \left\langle N^a_{t,\vec{x}} \Gamma_{t',\vec{x}'} \overline{N}^b_{0,\vec{0}} \right\rangle e^{-ikx'_z}$$

- Now take derivative w.r.t. k^2

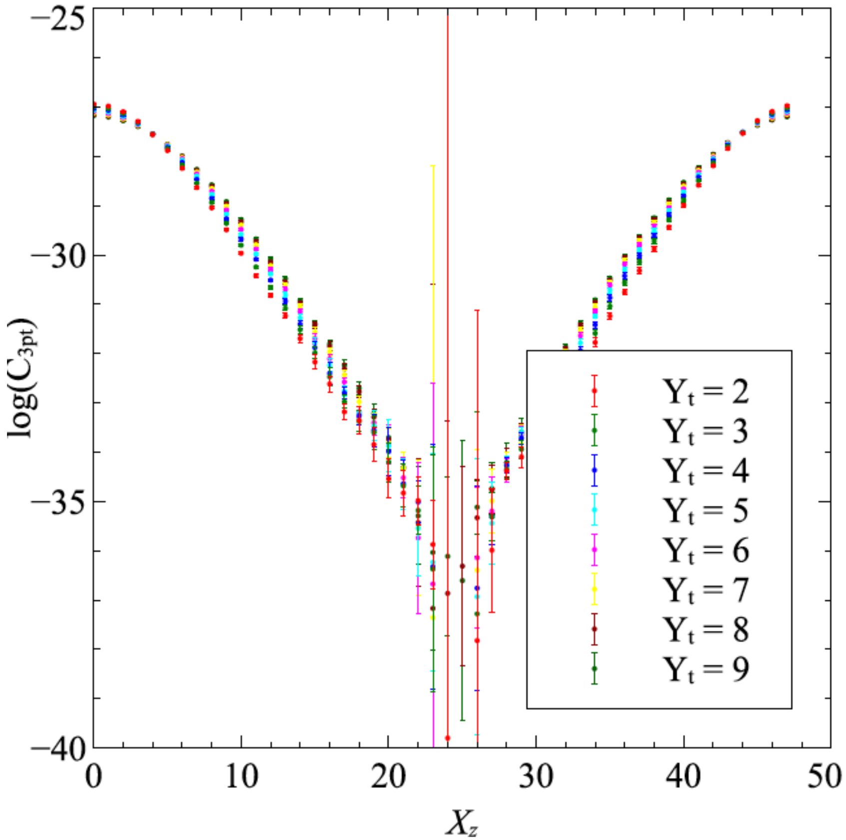
$$C'_{3\rm pt}(t,t') = \sum_{\vec{x},\vec{x}'} \frac{-x'_z}{2k} \sin\left(kx'_z\right) \left\langle N^a_{t,\vec{x}} \Gamma_{t',\vec{x}'} \overline{N}^b_{0,\vec{0}} \right\rangle$$

whence

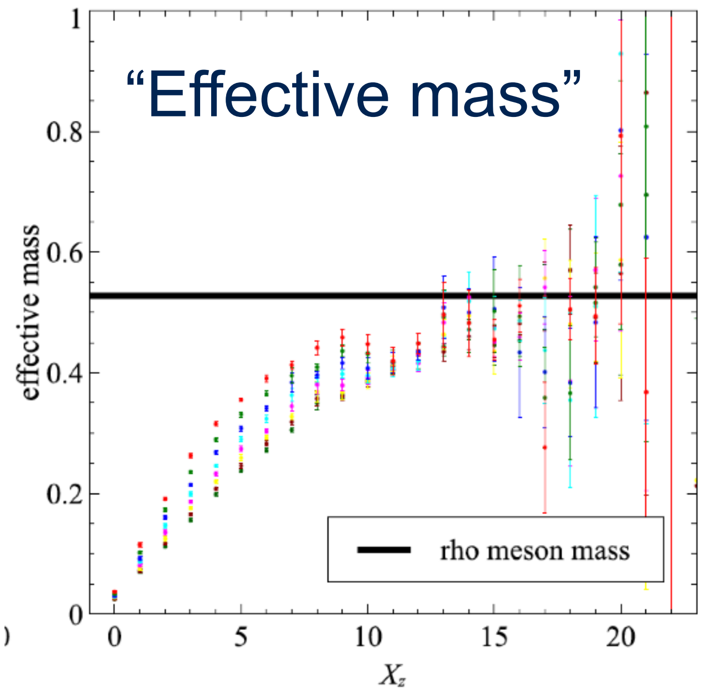
$$\lim_{k^2\to 0} C'_{3\rm pt}(t,t') = \sum_{\vec{x},\vec{x}'} \frac{-x'^2_z}{2} \left\langle N^a_{t,\vec{x}} \Gamma_{t',\vec{x}'} \overline{N}^b_{0,\vec{0}} \right\rangle .$$

Odd moments vanish by symmetry

Three-point correlator



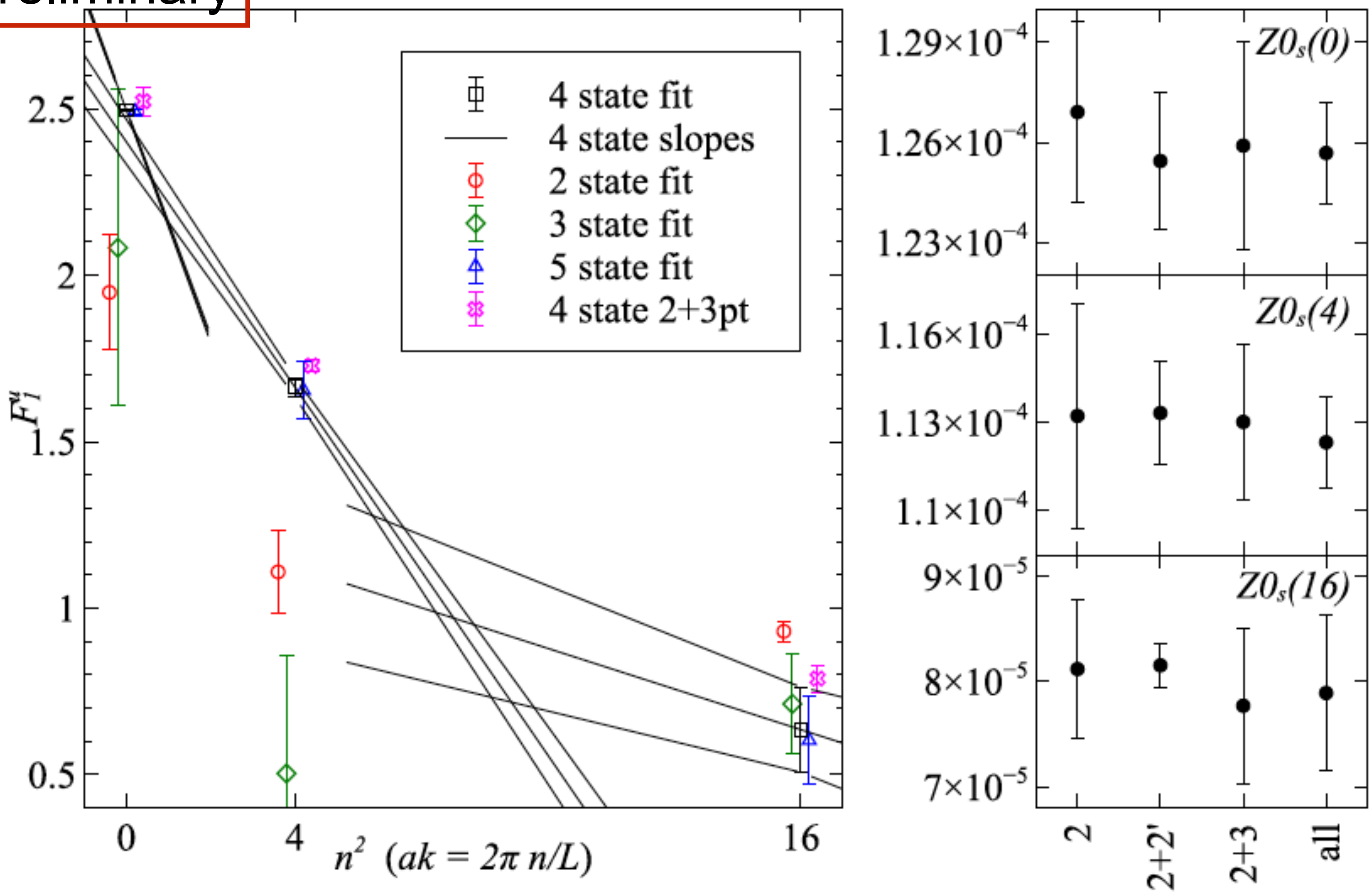
$\ln\left[C_{3\text{pt}}(t', x'_z)\right]$



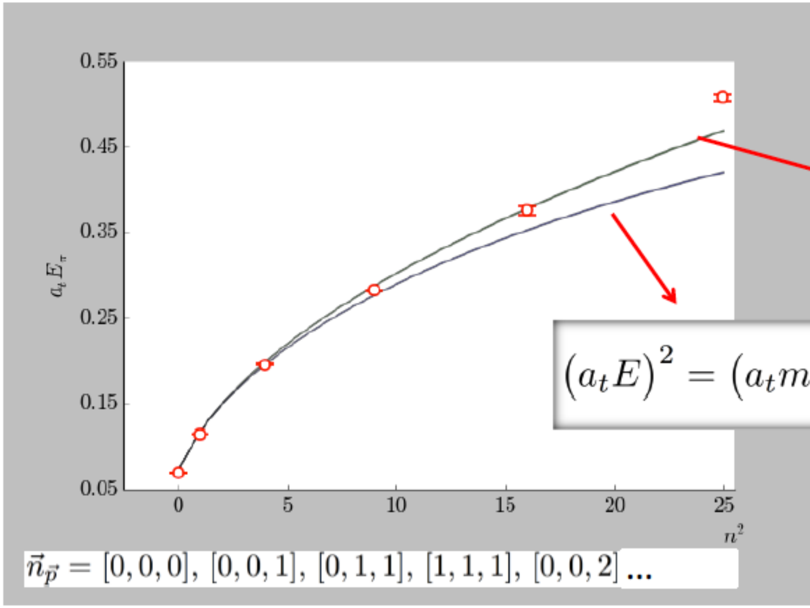
- Spatial moments push the peak of the correlator away from origin
- Larger finite volume corrections compared to regular correlators

F_1 Form Factor

Preliminary



Form Factor at High Momentum

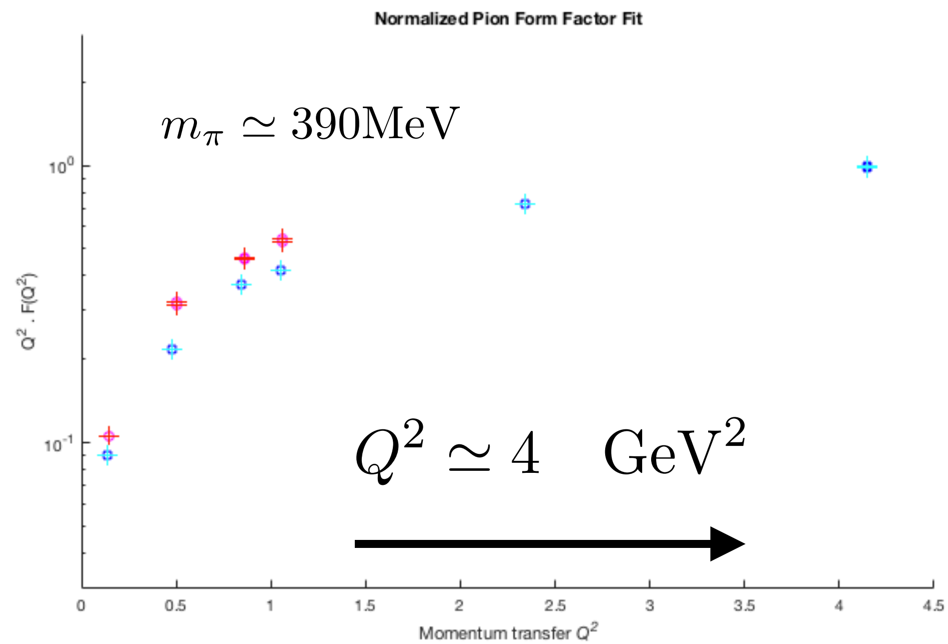


Dispersion relation:

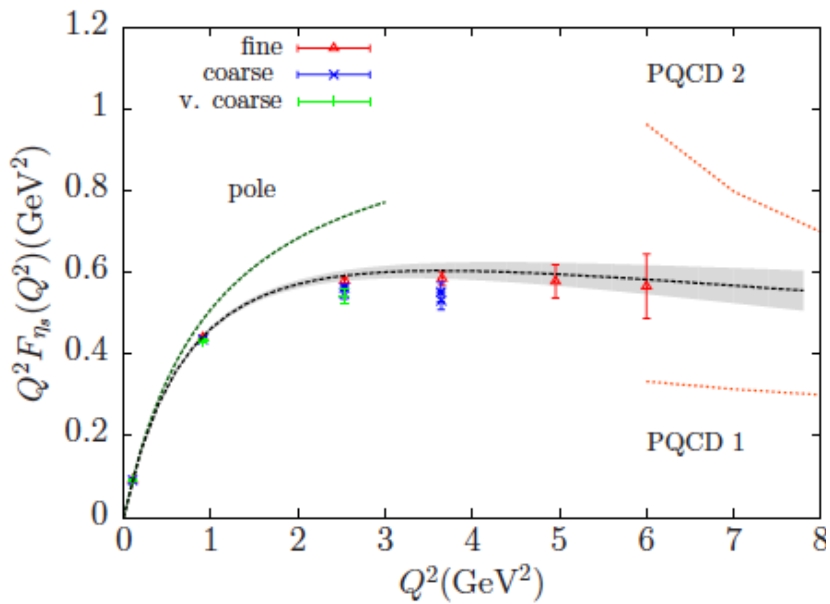
$$E^2 = m^2 + p^2$$

$$(a_t E)^2 = (a_t m)^2 + \left(\frac{2\pi}{\xi(L/a_s)}\right)^2 |\vec{n}_{\vec{p}}|^2$$

Very Preliminary!

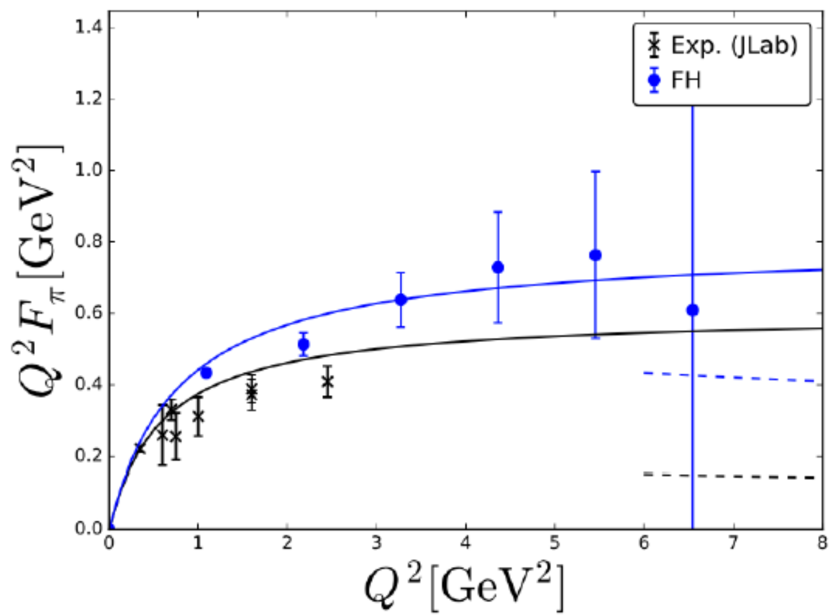


High Momentum - II



Momentum-smearing, at strange-quark masses

Koponen et al (HPQCD), PRD96, 054501 (2017)



Chambers et al (QCDSF/UKQCD/CSSM), arXiv: 1702.01513

Calculation using Feynman-Hellman Theory

$$H = H_0 + \lambda H_\lambda$$

$$\frac{\partial E_n}{\partial \lambda} = \langle n \mid H_\lambda \mid n \rangle$$

Quark Distribution Amplitude

Leading-twist EM form factor at high momentum governed by Quark Distribution Amplitude

$$\phi_\pi(x) = \frac{i}{f_\pi} \int \frac{d\xi}{2\pi} e^{i(x-1)\xi\lambda\cdot P} \langle \pi(P) \mid \bar{\psi}(0)\lambda\cdot\gamma\gamma_5\Gamma(0,\xi\lambda\psi(\xi\lambda) \mid 0 >$$

- *Generalized Parton Distributions (off-forward): GPDs*
- *PDFs*
- *(Transverse-Momentum-Dependent Distributions): TMDs*

- Euclidean lattice precludes the calculation of light-cone correlation functions
  - So… …Use *Operator-Product-Expansion* to formulate in terms of *local operators*

$$\mathcal{O}^{\mu_1\ldots\mu_n} = i^{n-1}\bar{\psi}\gamma_5\gamma^{\{\mu_1}D^{\mu_2}\ldots D^{\mu_n\}}\frac{\lambda^a}{2}\psi$$

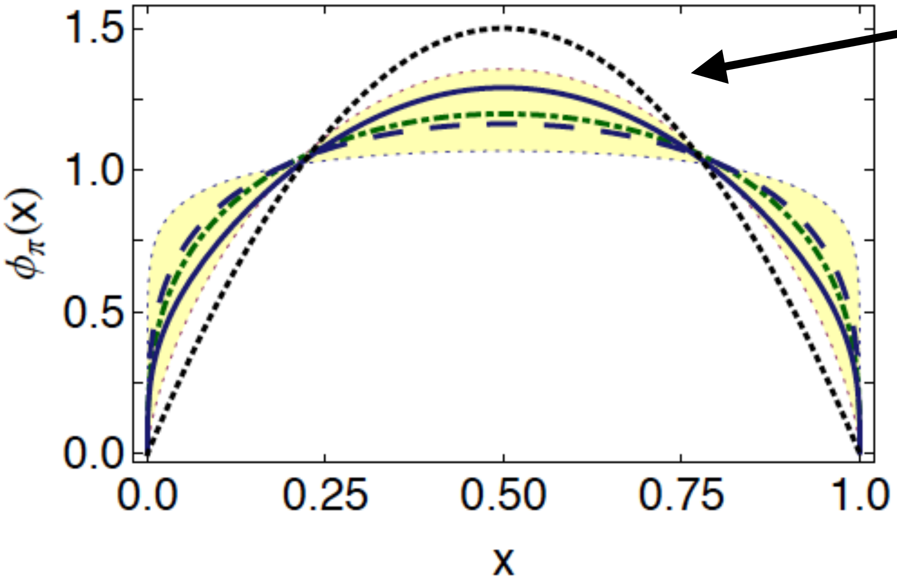
- Discretisation, and hence reduced symmetry of the lattice, introduces power-divergent mixing for N >3 moment.

QDA - II

- Lowest moment: f_π
- 1st moment: vanishes
- 2nd Moment: only constraint…

Martinelli, Sachrajda (87), Daniel, Gupta, Richards (91),…,Braun et al (2015)

Plot from Cloet et al, PRL111, 92001



Asymptotic form

Important for, e.g. DVMP

Quasi Distributions

- A solution, LaMET (Large Momentum Effective Theory) was proposed by X.Ji

X. Ji, Phys. Rev. Lett. 110 (2013) 262002

$$\tilde{\phi}(x,P) = \frac{i}{f_\pi}\int \frac{dz}{2\pi} e^{-i(x-1)Pz} \langle \pi(P) \mid \bar{\psi}(0)\gamma^z\gamma_5\Gamma(0,z)\psi(z) \mid 0\rangle$$

$$\tilde{\phi}(x,\Lambda,P) = \int_0^1 dy\, Z(x,y,\Lambda,\mu,P)\phi(y,\mu) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{P^2}, \frac{m_\pi^2}{P^2}\right)$$

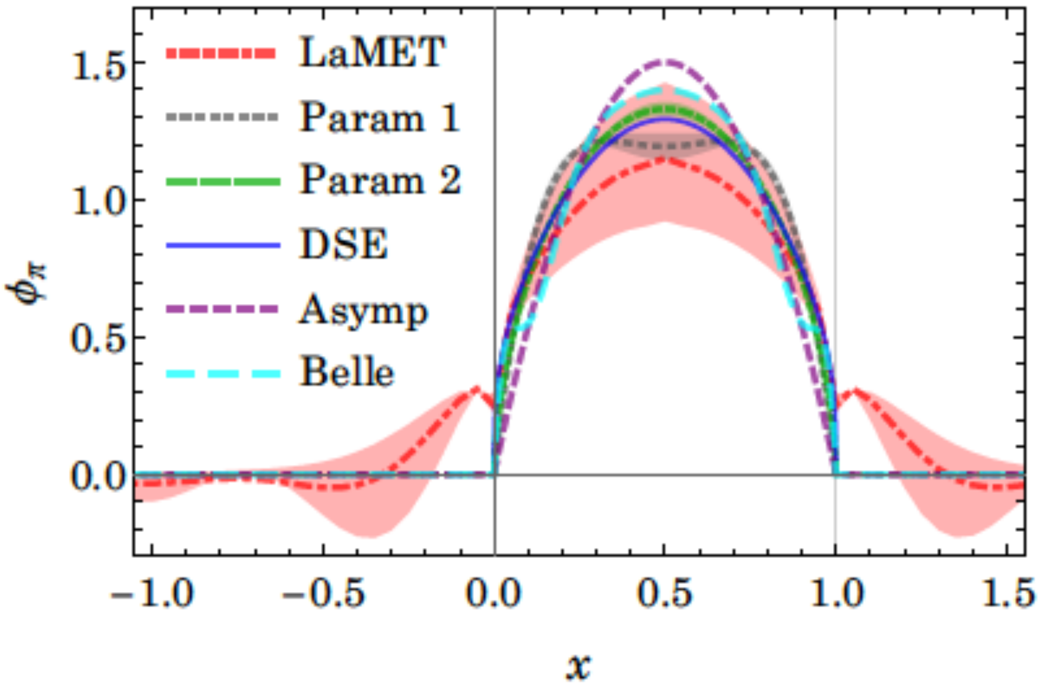
Y-Q Ma and J-W Qiu, arXiv:1404.6860, arXiv:1709.03018

- Matching and evolution of quasi- and light-cone distributions

Carlson, Freid, arXiv:1702.05775
Isikawa et al., arXiv:1609.02018
Monahan and Orginos, arXiv:1612.01584
Orginos, Radyushkin , et al arXiv:1706.05373 (**Pseudo Distributions**)
Briceno, Hansen, Monahan, arXiv:1703.06072 (Euclidean Signature)

Quasi-Distributions - II

Zhang et al., Phys. Rev. D 95, 094514 (2017)



Calculation in Ioffe time

Radyushkin, Phys. Rev. D 95, 056020 (2017)
Bali et al., arXiv:1709.04325

SUMMARY

- Pion Form Factor is key measure of transition from “soft” to “hard” degrees of freedom in QCD
- Simplest, and computationally least demanding, hadron for lattice structure calculations
- Self-contained confrontation for direct calculation of wave function, and twist expansion in lattice QCD?
- Operators and methods for quark distribution amplitudes provide exploratory theatre for studies of PDFs and GPDs in the nucleon.